Directed Search

Lecture 1: Introduction and Basic Formulations

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Main sources for this lecture:

- Shi, S., 2008, "Search Theory (New Perspectives),"
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  The New Palgrave Dictionary of Economics, 2nd edition,
  Palgrave, Macmillan.
- Burdett, K., S. Shi and R. Wright, 2001,
  "Pricing and Matching with Frictions," JPE 109, 1060-1085.
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  "Bidding for Labor," RED 3, 619-649.

1. Search Frictions and Search Theory

- Search frictions are prevalent:
  - unemployment, unsold goods, under-utilization
  - pervasive failure of the law of one price

- "Undirected search":
  individuals know the terms of trade only AFTER the match
  - bargaining: Diamond (82), Mortensen (82), Pissarides (90)
  - price posting: Burdett and Mortensen (98)

"Directed search":

- individuals choose what terms of trade to search for
- tradeoff between terms of trade and trading probability

Why should we care?

- prices should be important ex ante in resource allocation
- efficiency properties and policy recommendations
- robust inequality and unemployment
- tractability for analysis of dynamics and business cycles

Is directed search empirically relevant?

- Hall and Krueger (08):
  84% of white, non-college educated male workers either "knew exactly" or "had a pretty good idea" about how much their current job would pay at the time of the first interview.

- Holzer, Katz, and Krueger (91, QJE):
  (1982 Employment Opportunity Pilot Project Survey)
  firms in high-wage industries attract more applicants per vacancy than firms in low-wage industries after controlling for various effects.

Sketch of the lectures (if time permits):

- basic formulations of directed search
- matching patterns and inequality
- wage ladder and contracts
- business cycles
- monetary economics

2. Undirected Search and Inefficiency

One-period environment:

- workers: an exogenous, large number u
  - risk neutral, homogeneous
  - producing y when employed, 0 when unemployed

- firms/vacancies: endogenous number v
  - cost of a vacancy: $k \in (0, y)$
  - production cost $= 0$

Matching technology:

- matching function: $M(u,v)$ (constant returns to scale)
- tightness: $\theta = v/u$; matching probabilities:

$$\text{for a worker:} \quad p(\theta) = \frac{M(u,v)}{u} = M(1,\theta)$$
$$\text{for a vacancy:} \quad q(\theta) = \frac{M(u,v)}{v} = M(\tfrac{1}{\theta},1) = \frac{p(\theta)}{\theta}$$

- assumptions:
  $p(\theta)$ is strictly increasing and concave;
  $q(\theta)$ is strictly decreasing; $q(0)=1$, $q(\infty)=0$;
  worker's share of contribution to match:

$$s(\theta) \equiv \frac{u}{M}\,\frac{\partial M(u,v)}{\partial u} = 1 - \frac{\theta p'(\theta)}{p(\theta)} \quad \in [0,1]$$

Wage determination (Nash bargaining):

$$\max_{w \in [0,y]} \; w^{\sigma}(y-w)^{1-\sigma}, \quad \sigma\text{: worker's bargaining power}$$

$$\text{solution: } w = \sigma \, y$$

Equilibrium tightness:

- expected value of a vacancy:

$$J = q(\theta)(y-w) = (1-\sigma)q(\theta)y$$

- free entry of vacancies: $J = k$

$$\Longrightarrow w = y - \frac{k}{q(\theta)} \Longrightarrow q(\theta) = \frac{k}{(1-\sigma)y}$$

a unique solution for θ exists iff $0 < k < (1-\sigma)y$.

Social welfare and inefficiency:

- welfare function: $\mathcal{W} = u \times V + v \times (J - k) = u\, V$
- value for a worker:

$$V = p(\theta)w = p(\theta)\left[y - \frac{k}{q(\theta)}\right] = p(\theta)y - k\theta$$

- social welfare equals net output:

$$\mathcal{W} = u\, V = u\; p(\theta)y - (u\theta)k$$

- "constrained" efficient allocation:

$$\max_{\theta}\; \mathcal{W} = u\left[p(\theta)y - k\theta\right] \quad \Longrightarrow p'(\theta) = \frac{k}{y}$$

- rewrite the first-order condition for efficiency:

$$\frac{k}{y} = p'(\theta) = [1 - s(\theta)]\frac{p(\theta)}{\theta} = [1 - s(\theta)]q(\theta)$$

- compare with eqm condition, $\frac{k}{y} = (1 - \sigma)q(\theta)$: equilibrium is socially efficient if and only if

$s(\theta)$	$=$	σ
worker's share in creating match		bargaining power
Hosios (90) condition		

Why is this condition needed for efficiency?

- two externalities of adding one vacancy:
  - decreasing other vacancies' matching
  - increasing workers' matching
- internalizing the externalities:

private marginal value of vacancy	=	social marginal value of vacancy
$(y-w)q = (1-\sigma)qy$		$\frac{\partial M(u,v)}{\partial v}y = (1-s)qy$

  - if $1-\sigma > 1-s$, entry of vacancies is excessive
  - if $1-\sigma < 1-s$, entry of vacancies is deficient

Efficiency condition, $s(\theta) = \sigma$, is violated generically

- Cobb-Douglas: $M(u, v) = M_0 u^\alpha v^{1-\alpha}$

$$p(\theta) = M_0 \theta^{1-\alpha}, \quad s(\theta) = 1 - \frac{\theta p'(\theta)}{p(\theta} = \alpha \quad \text{(a constant)}$$

- telephone matching: $M(u, v) = \frac{uv}{u+v}$

$$p(\theta) = \frac{\theta}{1+\theta}, \quad s(\theta) = \frac{\theta}{1+\theta}$$

$$s(\theta) = \sigma \Longrightarrow \sigma = 1 - \left(\frac{k}{y}\right)^{1/2} \quad \text{(recall } p'(\theta) = \frac{k}{y}\text{)}$$

- urn-ball matching: $M(u, v) = v(1 - e^{-u/v})$

Cause of inefficiency:

> search is undirected: wage does not perform the role of allocating resources ex ante (before match)

- Nash bargaining splits the ex post match surplus
- it does not take matching prob into account

What about undirected search with wage posting?
(e.g., Burdett-Mortensen 98)

- similar inefficiency:
  workers cannot search for particular wages;
  workers receive all offers with the same probability

Criticisms on undirected search models:

- inefficiency arises from exogenously specified elements: Nash bargaining, matching function
- policy recommendations are arbitrary, depending on which way the efficiency condition is violated. E.g.
  - Should workers' search be subsidized?
- can we just impose the Hosios condition and go on?
  - fine for some analyses, but not useful if σ and the parameters in $s(\theta)$ change with policy

3. Directed Search and Efficiency

Directed search:

- Basic idea: individuals explicitly take into account the relationship between wage and the matching probability
- A more detailed description:
  - a continuum of "submarkets", indexed by w
  - market tightness function: $\theta(w)$
  - matching inside each submarket is random
  - matching probability:
    for a worker $p(\theta(w))$; for a vacancy: $q(\theta(w))$

Market tightness function: $\theta(w)$

- free entry of vacancies into each submarket
- complementary slackness condition for all w:

$$J(w) = q(\theta(w))(y - w) \leq k, \qquad \text{“} = \text{”} \text{ if } \theta(w) > 0$$

  - if there is potential surplus ($y - w > k$), then $J(w) = k$: firms are indifferent between such submarkets
  - if there is no potential surplus ($y - w \leq k$), then $\theta(w) = 0$

- solution:

$$\theta(w) = q^{-1}\left(\frac{k}{y-w}\right) \text{ whenever } w < y - k;$$

$$\theta(w) \text{ is strictly decreasing in } w$$

Worker's optimal search:
(This decision would not exist if search were undirected.)

- A worker chooses which submarket w to enter:

$$\max_w \; p(\theta(w))\, w \quad \text{where } \theta(w) = q^{-1}\left(\frac{k}{y-w}\right)$$

- tradeoff between wage w and matching prob $p(\theta(w))$:
  higher wage is more difficult to be obtained: $\frac{dp(\theta(w))}{dw} < 0$
- optimal choice:

$$w = -\frac{\tilde{p}(w)}{\tilde{p}'(w)}, \quad \tilde{p}(w) \equiv p(\theta(w))$$

Efficiency of directed search equilibrium:
Optimal directed search implies the Hosios condition:

$$\frac{w}{y} = s(\theta), \quad \text{where } s(\theta) = 1 - \frac{\theta p'(\theta)}{p(\theta)}$$

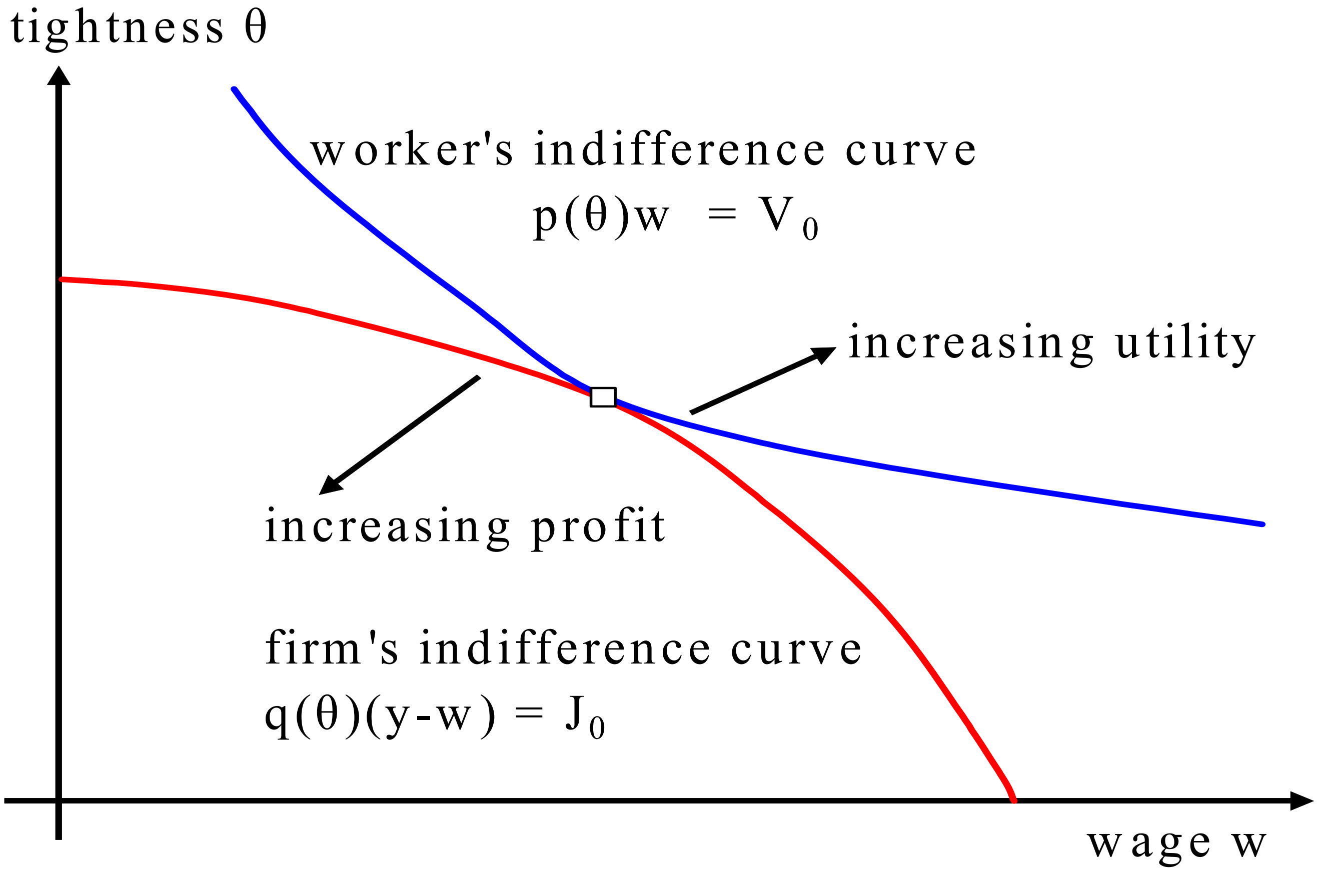
Proof:

$$\theta(w) = q^{-1}\left(\tfrac{k}{y-w}\right) \Longrightarrow \theta'(w) = \tfrac{q(\theta(w))/(y-w)}{q'(\theta(w))}$$
$$q(\theta) = \tfrac{p(\theta)}{\theta} \Longrightarrow \theta'(w) = \tfrac{\theta p(\theta)/(y-w)}{\theta p'(\theta)-p(\theta)}$$

$$\Longrightarrow w = -\frac{p(\theta)}{p'(\theta)\theta'(w)} = (\tfrac{p}{\theta p'} - 1)(y-w)$$
$$= (\tfrac{1}{1-s(\theta)} - 1)(y-w)$$

$$\Longrightarrow \frac{w}{y} = s(\theta). \quad \blacksquare$$

Hedonic pricing



4. Strategic Formulation of Directed Search

Motivation:

- The formulation above endogenizes the wage share;
  but the matching function is still a black box
- Is there a way to endogenize the mf as well?
- In a strategic formulation, total # of matches is
  an aggregate result of workers' application decisions
- some papers:
  Peters (91, ECMA),
  Burdett-Shi-Wright (01, JPE), Julien-Kennes-King (00, RED)

One-period game with directed search: BSW 01
(for fixed numbers u and v, for now)

- firms simultaneously post wages
- workers observe all posted wages
- each worker chooses which firm to apply to:
  no multiple applications
- each firm randomly chooses one among
  the received applicants to form a match

No coordination among firms or workers
$\Longrightarrow$ a worker and a vacancy may fail to match

Focus on symmetric equilibrium:

- all workers use the same strategy, including responses to a firm's deviation
- this implies that all firms post the same wage w

Why such a focus?

- tractability: in the case $u = v = 2$, there are many asymmetric equilibria which involve trigger strategies
- symmetric equilibrium emphasizes lack of coordination

A worker's strategy:
(when firm A posts x and other firms post w)

- each worker applies to firm A with probability a, and applies to each of the other firms with prob $\pi(a) = \frac{1-a}{v-1}$
- an applicant's indifference condition:

$$\underbrace{p(a)}_{\substack{\text{prob. of being}\\ \text{chosen by firm } A}} \; x \qquad = \qquad \underbrace{p(\pi(a))}_{\substack{\text{prob. of being}\\ \text{chosen elsewhere}}} \; w$$

- this solves $a = f(x, w)$:
workers' best response to firm A's deviation to x

A worker B's matching probability with firm A:

# of other app. to A	prob. of this event	conditional prob. that B is chosen
n	$C^n_{u-1} a^n (1-a)^{u-1-n}$	$\frac{1}{n+1}$

unconditional prob. that B matches with firm A:

$$\sum_{n=0}^{u-1} \frac{1}{n+1} C^n_{u-1} a^n (1-a)^{u-1-n} = \sum_{n=0}^{u-1} \frac{(u-1)!\ a^n (1-a)^{u-1-n}}{(n+1)!(u-1-n)!}$$

$$= \frac{1}{ua} \sum_{n=1}^{u} \frac{u!}{n!(u-n)!} a^n (1-a)^{u-n} = \frac{1-(1-a)^u}{ua} \quad (\equiv p(a))$$

Firm A's optimal choice:

- queue length (expected #) of applicants to firm A:

$$\sum_{n=1}^{u} n\ C_u^n a^n (1-a)^{u-n} = \sum_{n=1}^{u} \frac{u!\ a^n (1-a)^{u-n}}{(n-1)!(u-n)!}$$
$$= ua \sum_{n=0}^{u-1} \frac{(u-1)!}{n!(u-n)!} a^n (1-a)^{u-1-n} = ua.$$

- tightness for firm A, $\frac{1}{uf(x,w)}$, is indeed a function of x
- firm A's matching probability:

$$\sum_{n=1}^{u} C_u^n a^n (1-a)^{u-n} = 1-(1-a)^u$$

Firm A's optimal choice:

- choosing wage $x = g(w)$ to solve:

$$\max_{(x,a)} \left[1 - (1-a)^u\right](y - x)$$

$$\text{s.t.} \quad \frac{1-(1-a)^u}{ua} x = \frac{1 - [1-\pi(a)]^u}{u\pi(a)} w$$

- tradeoff with a higher x:
  - lower ex post profit $(y - x)$
  - higher matching probability $[1 - (1-a)^u]$:
    - $a = f(x, w)$ satisfies the constraint;
    - it is an increasing function of x

Symmetric equilibrium:

$$\text{wage } w \text{ that satisfies } w = g(w).$$

- worker's application prob.: $a = \pi(a) = \dfrac{1}{v}$
- queue length for each firm: $ua = \dfrac{u}{v} = \dfrac{1}{\theta}$
- firm's matching probability:

$$q(u, v) = 1 - (1-a)^u = 1 - (1 - \frac{1}{v})^u$$

- firm's first-order condition yields:

$$w = y\left[\frac{(1 - 1/v)^{-u} - 1}{u/v} - \frac{1}{v-1}\right]^{-1}$$

Why does this equilibrium look different?

- endogenous matching function:

$$M(u,v) = v\ q(u,v) = v\left[1-(1-\frac{1}{v})^u\right]$$

  – decreasing returns to scale:

$$q(2u,2v) < q(u,v) \Longrightarrow M(2u,2v) < 2M(u,v)$$

  – coordination failure is more severe
  when there are more participants on each side

- deviating firm can affect a worker's payoff elsewhere:

$$\frac{1-[1-\pi(a)]^u}{u\pi(a)}w, \quad \text{where } \pi(a) = \frac{1-a}{v-1}$$

All works out well in the limit $u, v \to \infty$:
[denote $\theta = \lim \frac{v}{u} \in (0, \infty)$]

- constant returns to scale in matching:

$$\begin{aligned} q(u,v) &= 1 - (1 - \tfrac{1}{v})^u \\ &= 1 - (1 - \tfrac{1}{\theta u})^u \to 1 - e^{-1/\theta} \end{aligned}$$

$$p(u,v) = \frac{1 - (1 - \frac{1}{v})^u}{u/v} \to \theta \left(1 - e^{-1/\theta}\right)$$

- a firm's deviation no longer affects the queue length of applicants elsewhere:

$$u\pi(a) = u\frac{1-a}{v-1} \to \frac{1}{\theta}$$

The limit $u, v \to \infty$: (continued)

- equilibrium wage share satisfies Hosios condition:

$$\frac{w}{y} = \left[\frac{(1-1/v)^{-u}-1}{u/v} - \frac{1}{v-1}\right]^{-1}$$
$$\to \frac{1}{\theta[e^{1/\theta}-1]} = 1 - \frac{\theta p'(\theta)}{p(\theta)} \equiv s(\theta)$$

recall: $p(\theta) = \theta(1 - e^{-1/\theta}), \quad q(\theta) = 1 - e^{-1/\theta}$

- expected payoff equals the expected social value:

a worker: $pw \to y\ e^{-1/\theta}$

a firm: $q(y - w) \to y \left[1 - (1 + \frac{1}{\theta})e^{-1/\theta}\right]$

Explain eqm expected payoff as social marginal values:

- A worker's expected payoff

$$pw = y \times \underbrace{e^{-1/\theta}}_{\text{prob. that a firm fails to match}}$$

Adding a worker to match with a firm creates social value only when the firm does not have a match.

- A firm's expected payoff

$$q(y-w) = y \underbrace{\left(1 - e^{-1/\theta}\right)}_{\substack{\text{firm's matching} \\ \text{probability}}} - y \underbrace{\frac{1}{\theta} e^{-1/\theta}}_{\substack{\text{crowding-out} \\ \text{on other firms}}}$$

Equilibrium tightness in the limit $u, v \to \infty$:

- free entry of vacancies implies: $q(y - w) = k$

$$\text{i.e.} \qquad \underbrace{1 - (1 + \frac{1}{\theta})e^{-1/\theta}}_{\text{strictly decreasing in } \theta} = \frac{k}{y}$$

- for any $k \in (0, y)$, there is a unique solution $\theta \in (0, \infty)$

A game with first-price auctions: JKK 00
(for fixed numbers u and v)

- firms post auctions with reserve wages above which a firm does not hire a worker
- workers observe all posted reserve wages
- each worker chooses which firm to apply to
- after receiving a number $n \geq 1$ of applicants:
  - if $n \geq 2$, the applicants bid in first-price auction (i.e., the worker with the lowest wage offer wins)
  - if $n = 1$, the worker is paid the reserve wage

Consider firm A that posts reserve wage x
(while all other firms post reserve wage r)

- each worker visits firm A with prob. $a = f(x, r)$
- payoff to a worker (B) who visits firm A:

# of other visitors, n	prob. of the event	worker B's payoff
$n = 0$	$(1-a)^{u-1}$	x
$n \geq 1$	$1-(1-a)^{u-1}$	0

- $a = f(x, r)$ solves a worker's indifference condition:

$$(1-a)^{u-1}x = [1-\pi(a)]^{u-1}\, r, \text{ where } \pi(a) = \frac{1-a}{v-1}$$

- payoff to firm A:

# of visitors, n	prob. of the event	payoff
$n = 1$	$ua(1-a)^{u-1}$	$y - x$
$n \geq 1$	$1-(1-a)^u - ua(1-a)^{u-1}$	y

- firm A's optimal choice of x:

$$\max_{(x,a)} ua(1-a)^{u-1}(y-x) + \left[1-(1-a)^u - ua(1-a)^{u-1}\right] y$$

$$\text{s.t. } (1-a)^{u-1}x = [1-\pi(a)]^{u-1}\, r$$

- solution (firm A's best response to other firms): $x = g(r)$

Symmetric equilibrium: $r = g(r)$

- the limit when $u, v \longrightarrow \infty$:
  - queue length: $ua = u/v \longrightarrow 1/\theta$
  - reserve wage: $r \longrightarrow y$
  - equilibrium wage distribution:

wage	prob
y	$(1-a)^{u-1} \longrightarrow e^{-1/\theta}$
0	$1-(1-a)^{u-1} \longrightarrow 1-e^{-1/\theta}$

- equivalence to wage posting in expected payoff:

$$\text{a worker: } y\ e^{-1/\theta}; \quad \text{a firm: } y\left[1-(1+\frac{1}{\theta})e^{-1/\theta}\right]$$

General lessons:

- directed search makes sense:
  ex ante tradeoff between terms of trade and probability
- directed search can attain constrained efficiency
  in the canonical search environment
- the mechanism to direct search is not unique:
  price/wage posting, auctions, contracts
  - commitment to the terms of trade is the key
  - uniform price is not necessary for efficiency
    when agents are risk-neutral