Directed Search

Lecture 3: Wage Ladder and Contracts

Lectures at Osaka University (2012)

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Main sources for this lecture:

- Shi, S., 2009, "Directed Search for Equilibrium Wage-Tenure Contracts," ECMA 77, 561-584.
- Delacroix, A. and S. Shi, 2006, "Directed Search On the Job and the Wage Ladder," IER 47, 651-699.
- Tsuyuhara, K., 2010, "Dynamic Contracts with Worker Mobility via Directed On-the-Job Search," manuscript.
- Menzio, G. and S. Shi, 2011, "Efficient Search on the Job and the Business Cycle," JPE 119, 468-510.

1. Motivation

Facts:

- job-to-job transition is frequent in a worker's career:
  - 2.6% of employed workers change employers per month (Fallick and Fleischman 04)
  - average # of jobs = 7 in first 10 years (Topel and Ward 92)
- wage is a key determinant of mobility (Farber 99):
  - wage increases with tenure
  - high-wage workers are less like to quit
- limited mobility and wage ladder (Buchinsky and Hunt 99):
  - most of wage movements are between adjacent quintiles

Some explanations:

- learning about productivity:
  Jovanovic (79), Harris and Holmstrom (82),
  Moscarini (05), Gonzalez and Shi (00)

- match-specific productivity and heterogeneity:
  Postel-Vinay and Robin (02), Burdett and Coles (06)

These explanations are useful, but not enough to explain:

- residual wage inequality

- wage ladder and limited wage mobility

On-the-job search (OJS) may be important for these facts:

- Burdett and Mortensen (98):
  - posting of wage levels + OJS $\Longrightarrow$ wage dispersion among homogeneous workers
  - key insights:
    luck in search $\Longrightarrow$ heterogeneous search outcomes
    $\Longrightarrow$ heterogeneous outside options in further search
    $\Longrightarrow$ continuous non-degenerate wage distribution
- Burdett and Coles (03, **BC**):
  extend to wage-tenure contracts + OJS $\Longrightarrow$
  wage rises and quit rate falls with tenure

Search is undirected in BC (03):

- does not capture the wage ladder:
  - all applicants draw offer from the same distribution
  - have the same prob. of moving to the highest wage
- robustness issue:
  do wage dispersion and the tenure effect depend on the assumption that applicants do not know offers ex ante?
- tractability: analysis is complicated because the wage distribution affects decisions as a state variable

Directed search:

- makes sense in terms of economics
- OJS is likely to be directed (referral, etc.)
- robust wage dispersion and tenure effect

Why is directed search hopeful of producing a wage ladder?

- workers at different wages differ in reservation values
- they choose to search for different values:
  - high-wage workers search for higher values
  - climb up the wage ladder

2. Model Environment (in Continuous Time)

Workers:

- continuum with measure one;
  rate of time preference: ρ;   death rate: δ
  effective discount rate $r = \rho + \delta$
  identical productivity: y;   unemployment benefit: b
- for contracts to be interesting, workers are assumed to be
  - risk averse: $u''(w) < 0$ (and $u'(0) = \infty$)
  - not able to borrow against future income
- employed worker can search on the job at rate $\lambda_1 > 0$;
  unemployed worker can search at rate $\lambda_0 > 0$

Firms:

- risk neutral
- each firm hires one worker
- number of vacancies is determined by free entry;
  flow cost of a vacancy $= k > 0$
- identical firms:
  cost of production $= 0$;   output $= y$

Wage-tenure contracts (offered at time s):

$$W(s) = \{\tilde{w}(t, s)\}_{t=0}^{\infty}$$

- tenure t: $t = \varnothing$ is "tenure" of unemployed worker

- value of a contract (discounted sum of utilities to a worker):
  $V(0, s) = x$: an offer at s;
  $V(t, s)$: continuation value from time $(t + s)$ onward;
  bounds: $V \in \left[\underline{V}, \bar{V}\right]$

$$\underline{V} = \frac{u(b)}{r}, \quad \bar{V} = \frac{u(\bar{w})}{r}$$

  $\bar{w}$: highest wage, to be determined later

Assumptions on contracts:

- a worker can quit at any time

- a firm commits to the contract

- firms do not respond to employee's outside offers

Examples without the last assumption:

Harris and Holmstrom (82), Postel-Vinay and Robin (02)

Directed search:

- treat different offers x as different submarkets
- workers and firms choose which submarket to enter
- submarket x: # of vacancies $= N(x)$
  - tightness $\theta(x)$: applicants/firms ratio
  - Poisson rate of matches: $\mathcal{M}\left(N(x), \frac{N(x)}{\theta(x)}\right)$
  - matching rates for participants:

$$\text{for a vacancy: } q(x) = \mathcal{M}(\theta(x), 1)$$

$$\text{for a worker: } p(x) = \mathcal{M}(\theta(x), 1)/\theta(x)$$

Matching function:

$$q(x) = \mathcal{M}(\theta(x), 1), \quad p(x) = \frac{\mathcal{M}(\theta(x),1)}{\theta(x)}$$

$$\xrightarrow{\text{eliminate } \theta} \qquad p(x) = M(q(x))$$

- refer to $M(.)$ as the matching function, which is exogenous and implied by $\mathcal{M}$
- but $\theta(.)$, $p(.)$ and $q(.)$ are all endogenous
- look for equilibrium with $p' < 0$ and $p'' < 0$.

- Assumptions on matching function M:

  (i) continuous for all $q \in [\underline{q}, \bar{q}]$, with $\bar{q} < \infty$

  (ii) $M' (q) < 0$, with $M (\bar{q}) = 0$ (i.e., $p \left(\bar{V} \right) = 0$)

  (iii) twice differentiable, with bounded $\left| M' \right|$ and $\left| M'' \right|$

  (iv) $qM'' (q) + 2M' (q) \leq 0$.

- An example: $\mathcal{M}(\theta, 1) = [\alpha\theta^{\sigma} + 1 - \alpha]^{1/\sigma}$

$$\Longrightarrow \ p = \hat{M}(q) = \left[\frac{1 - (1-\alpha)\, q^{-\sigma}}{\alpha}\right]^{-1/\sigma}$$

$\sigma = -1$: all assumptions are satisfied;
for $\sigma \geq 0$: set $\bar{q} < \infty$, and $M(q) = \hat{M}(q) - \hat{M}(\bar{q})$
for $\sigma < 0$ (and $\sigma \neq -1$): let

$$q_0 = (1-\epsilon)(1-\alpha)^{1/\sigma}, \quad \bar{q} = q_0 - \frac{\hat{M}(q_0)}{\hat{M}'(q_0)}$$

$$M(q) = \begin{cases} \hat{M}(q), & \text{if } q \leq q_0 \\ \hat{M}(q_0) + \hat{M}'(q_0)(q - q_0), & \text{if } q_0 < q \leq \bar{q} \end{cases}$$

3. Optimal Decision
Optimal decision: application

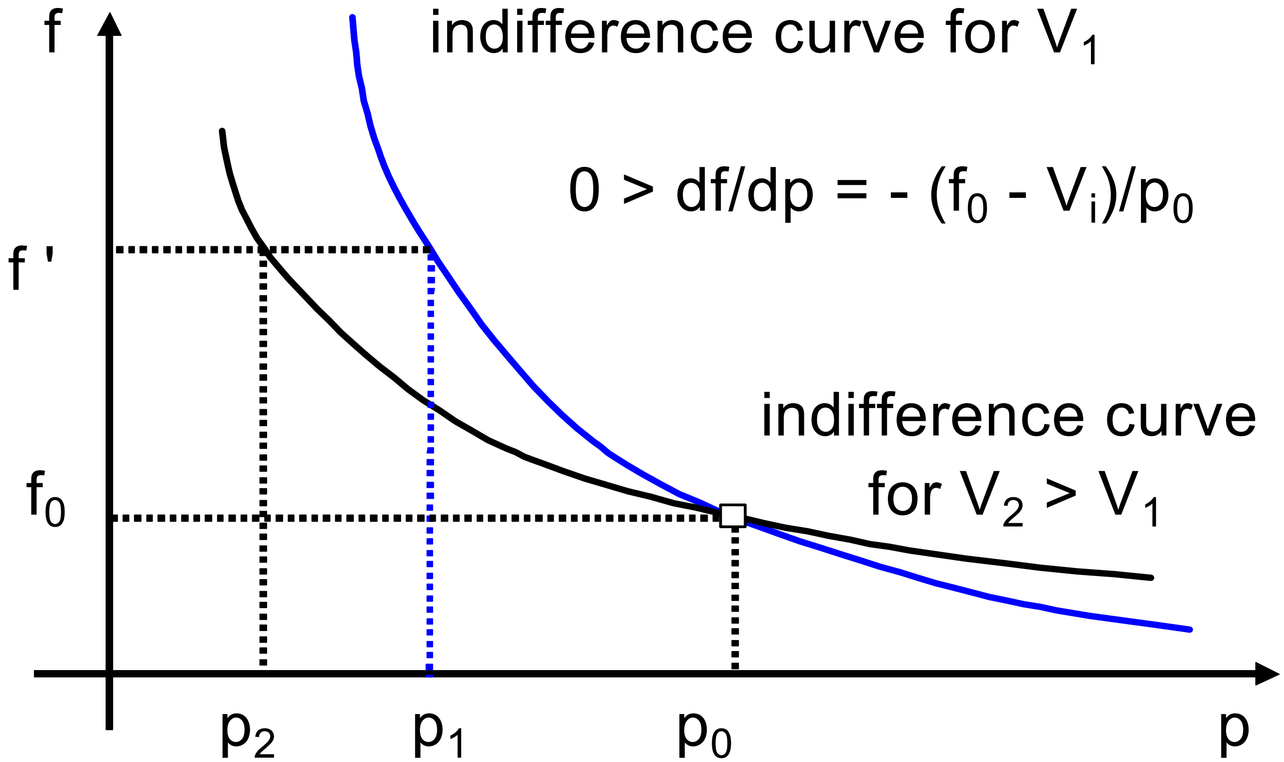
(This decision does not exist if search is undirected.)

- a worker whose current value is $V(t)$ solves:

$$S(V(t)) \equiv \max_{x \in [V(t), \bar{V}]} p(x)\,[x - V(t)]$$

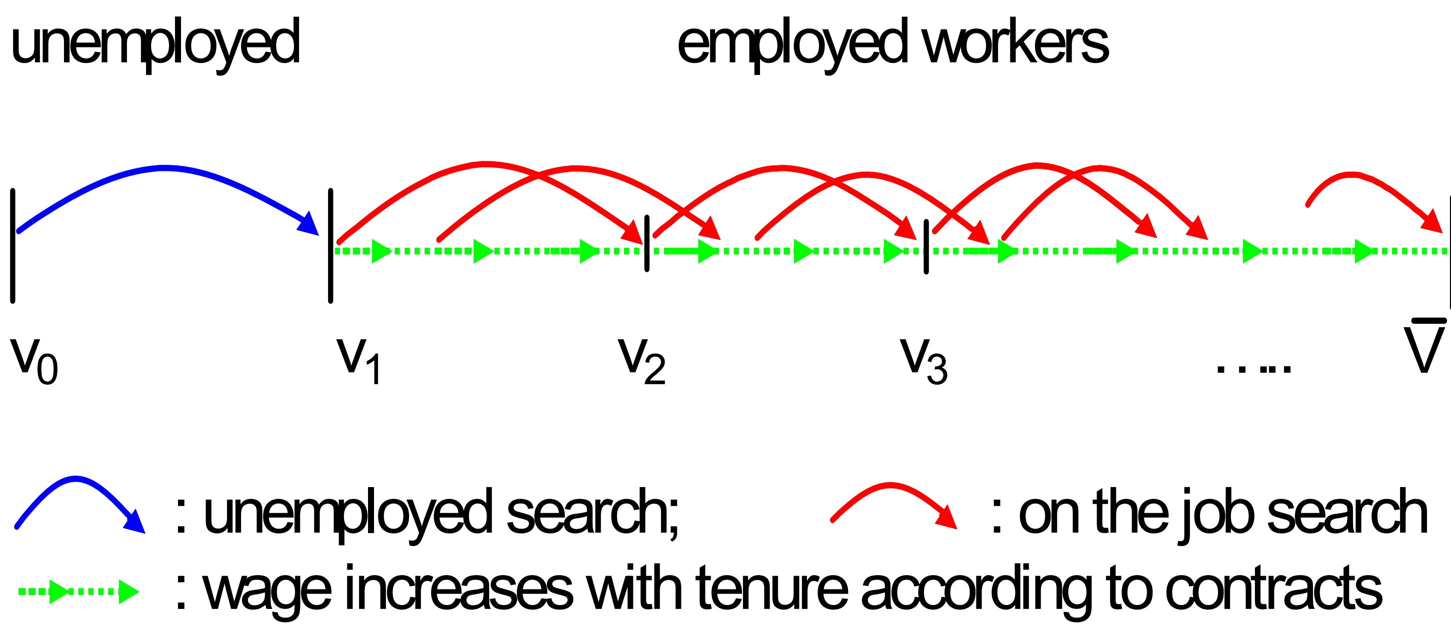
- tradeoff: probability $p(x)$ and gain $[x - V(t)]$
- optimal choice $F(V) \equiv \arg\max p(x)\,[x - V]$:
  - unique for each V ($\Longrightarrow$ endogenous separation)
  - increasing: $F'(V) > 0$ (ladder)
  - diminishing gains: $[F(V) - V]$ and $S(V)$ decrease in V

$$S(V(t)) \equiv \max\ p(f)\,[f - V(t)]$$



Single-crossing property

Implied career path of a worker:



$$v_0 = V_u, \quad v_j = F^{(j)}(v_0),\ j = 1, 2, \ldots$$

Value functions:

- employed worker with tenure t:

$$\underset{\text{"permanent income"}}{\rho V(t)} = \underset{\text{utility + gain from increase in tenure}}{u(\tilde{w}(t)) + \frac{dV(t)}{dt}} + \underset{\text{gain from search}}{\lambda_1 S(V(t))} - \delta V(t)$$

$$\Longrightarrow \frac{dV(t)}{dt} = rV(t) - \lambda_1 S(V(t)) - u(\tilde{w}(t)), \quad (r = \rho + \delta)$$

- unemployed worker (with $t = \varnothing$):

$$0 = rV_u - \lambda_0 S(V_u) - u(b)$$

(unemp. benefit does not change over duration)

Value functions (continued):

- firm that has a worker with tenure t:

$$\frac{d\tilde{J}(t)}{dt} = \left[r + \underbrace{\lambda_1 \, p\left(F\left(V\left(t\right)\right)\right)}_{\text{worker's endogenous separation rate}}\right] \tilde{J}\left(t\right) - \left[y - \tilde{w}(t)\right]$$

- integrate:

$$\tilde{J}(t_a) = \int_{t_a}^{\infty} \left[y - \tilde{w}(t)\right] \gamma(t, t_a) dt$$

$$\text{where } \gamma(t, t_a) \equiv \exp\left[-\int_{t_a}^{t} \left[r + p(F(V(\tau)))\right] d\tau\right]$$

Recruiting decision at time s:

- two parts of the decision:
  - part 1: optimal offer $x = V(0)$ to maximize $q(x)\tilde{J}(0)$
  - part 2: contract to deliver $V(0)$ and maximize $\tilde{J}(0)$

- part 1: choose the offer $x = V(0)$
  - tradeoff between prob q and value $\tilde{J}$
  - eqm $q(.)$ is such that a firm is indifferent among a continuum of offer values x such that $q\left(x\right)\tilde{J}\left(0\right) = k$
  - implied bounds on value and wage:

$$q\left(\bar{V}\right)\underline{J} = k \quad \Longrightarrow \bar{w} = y - rk/\bar{q}, \quad \underline{J} = k/\bar{q}$$

• part 2: given $V(0)$, optimal contract $\{\tilde{w}(t)\}_{t\geq 0}$ solves

$$(\mathcal{P}) \;\; \max \;\; \tilde{J}(0) = \int_0^{\infty} [y - \tilde{w}(t)]\,\gamma(t,0)dt$$

subject to

$$\frac{dV(t)}{dt} = rV(t) - \lambda_1 S(V(t)) - u\left(\tilde{w}\left(t\right)\right), \quad V\left(0\right) = x$$

$$\frac{d}{dt}\gamma\left(t,0\right) = -\left[r + p(F(V(t)))\right]\gamma(t,0)$$

Solve this problem with the Hamiltonian:

$$\begin{aligned}\mathcal{H}(t) = &\left(y - \tilde{w}\right)\gamma(t,0) + \Lambda_V\left[rV - S(V) - u(\tilde{w})\right] \\ &-\Lambda_\gamma\left[r + p\left(F\left(V\right)\right)\right]\gamma(t,0)\end{aligned}$$

Properties of optimal contracts:

- wage and value increase with tenure:

$$0 < \frac{d\tilde{w}}{dt} = \underbrace{-\frac{[u'(\tilde{w})]^2}{u''(\tilde{w})}}_{\substack{\text{risk}\\ \text{aversion}}} \times J(V) \times \underbrace{\lambda_1 \left[-\frac{d}{dV} p(F(V)) \right]}_{\substack{\text{backloading wages}\\ \text{to reduce quit}}}$$

two considerations:

– backloading wages to reduce incentive to quit

– risk aversion: making backloading smooth
(if workers are risk neutral, wage jumps are possible)

Properties of optimal contracts (continued):

- values for workers increase with tenure:

$$\dot{V}(t) > 0, \quad \text{all } t < \infty$$

  if $V(t)$ has a decreasing segment, replacing it with a constant reduces quit rate and increases firm value

- efficient sharing of value between firm and worker:

$$-\dot{J}(t) = \frac{\dot{V}(t)}{u'(\tilde{w}(t))}$$

  a dollar value given up by a firm is gained by the worker

Properties of optimal contracts (continued):

- wages are positive ($\tilde{w}(t) > 0$ for all t) if

$$\lambda_1 - \lambda_0 \leq [u(b) - u(0)] / S(\underline{V})$$

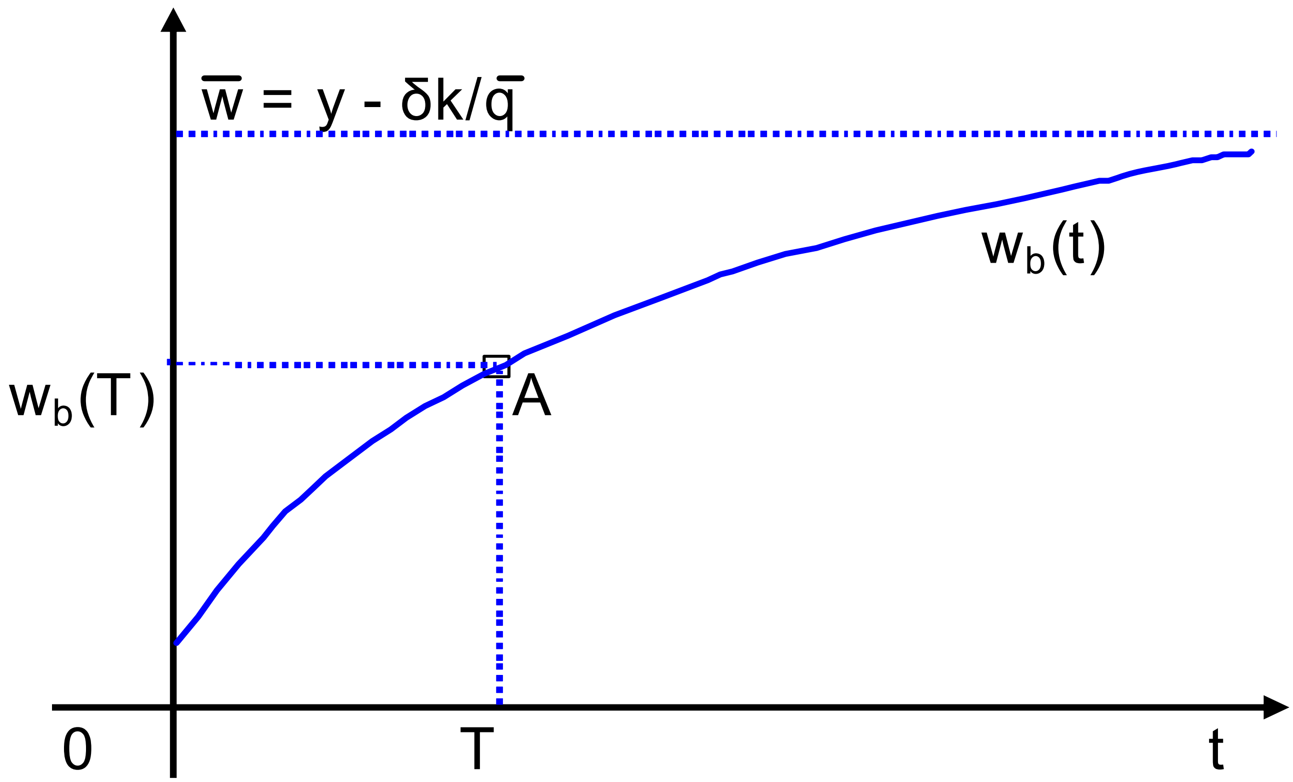
  in fact, $\tilde{w}(t) \geq b$ if $\lambda_1 \leq \lambda_0$:

  – an unemployed worker can apply to all the offers that an employed worker can apply

  – if $\tilde{w}(t) < b$, an employed worker is better offer quitting into unemployment and search

- set of optimal contracts: segments of a "baseline contract"

Baseline contract



$$\text{all contracts } \{\tilde{w}(t)\}_{t=0}^{\infty}: \quad \tilde{w}(t) = \tilde{w}_b(t+s), \text{ some } s \geq 0$$

Use the baseline contract to describe the equilibrium:

- set of equilibrium offers: $\mathcal{V} = \{V(t) : t \geq 0\}$
  $V(0)$: initial value of baseline contract

- $T(z)$: length of time taken to reach value z according to the baseline contract; i.e., $V(T(z)) = z$

- change notation from wage to value:

$$w(V) = \tilde{w}(T(V)) : \text{ wage function}$$
$$J(V) = \tilde{J}(T(V)) : \text{ firm value}$$

4. Equilibrium Definition:

A stationary equilibrium consists of two blocks:

block 1: $[\mathcal{V}, p(V), q\,(V), F(V), w(V), J(V)]$ that satisfy

- (i) optimal application: $F(V)$, given $p(.)$
- (ii) optimal contracts and values:
  each value $V \in \mathcal{V}$ is delivered by an optimal contract, starting with wage $w(V)$ and generating firm value $J(V)$
- (iii) $p\,(.)$ and $q\,(.)$ satisfy: $p(V) = M(q(V))$ and

$$
\begin{aligned}
q(V)J(V) &= k \quad \textbf{for all } V \in [\underline{V}, \bar{V}] \quad \text{(free entry)} \\
&< k \quad \text{for all } V \notin [\underline{V}, \bar{V}]
\end{aligned}
$$

block 2: distribution of workers, G, that satisfies

- (iv) G is stationary

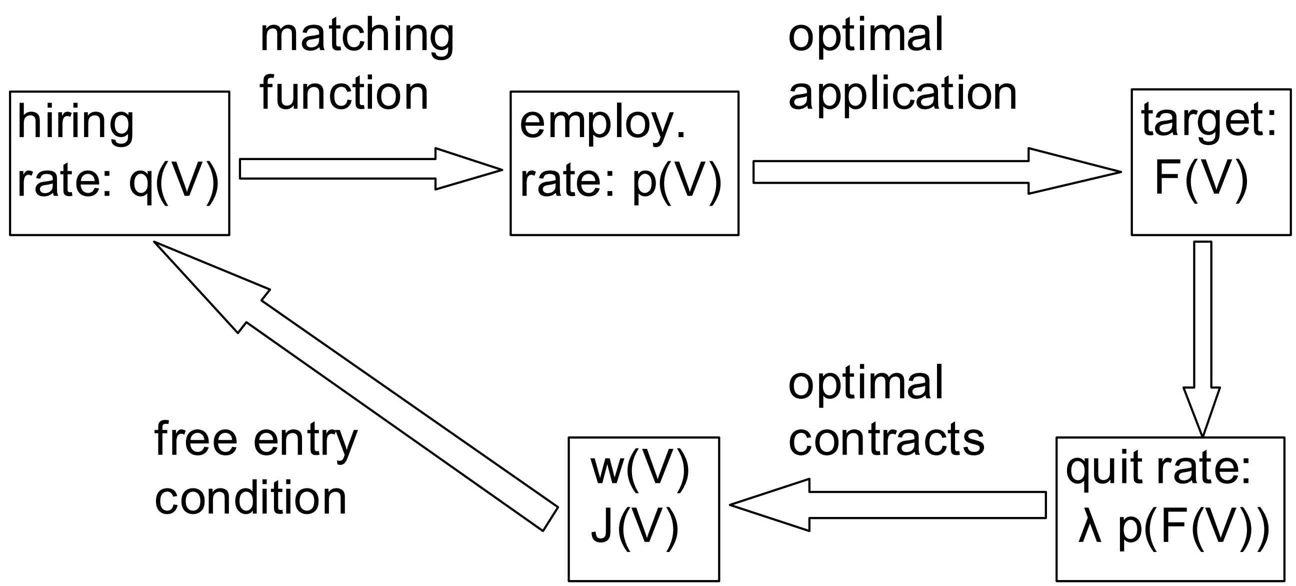
Block recursivity (BR):

- block 1 is independent of block 2:
  - we can solve eqm values, contracts and matching prob functions WITHOUT any reference to the distribution
  - this reduces the state space significantly $\Longrightarrow$ tractability
- then we compute distribution by counting worker flows (distribution is still important for the aggregates).

Directed search and free entry are important for BR:

- directed search $\Longrightarrow$ endogenous separation:
  workers at different values optimally choose to
  search for different future values
- applicants care only about the submarket they search:
  - not about how many workers search in other submarkets
  - if matching rate functions do not depend on G, then
    contracts, values and search decision do not depend on G
- free entry of vacancies $\Longrightarrow$
  matching rate functions are indeed independent of G

Why is the equilibrium block recursive?



Fixed-point problem

Why does block recursivity fail with undirected search?

- case in which wage-tenure contracts are posted (BC 03):
  - worker who receives a firm's offer is randomly drawn;
    $\Longrightarrow$ acceptance prob depends on where the worker is in G
  - current worker's quit prob depends on offer distribution
    $\Longrightarrow$ firm value and contracts depend on the distribution
- case in which wage-tenure contracts are bargained:
  - worker's outside option in bargaining is a random draw from the distribution $\Longrightarrow$ contracts depend on distribution

Contrasts between the two search frameworks

Directed search:	Undirected search:
(i) optimal application: given $p(.)$, chooses $F(V)$;	(i) random application: $\lambda_1 \left[1 - O(V)\right];$ O: offer distribution
(ii) hiring rate: $q(V) = M^{-1}(p(V));$	(ii) hiring rate: $q\left(V\right) \equiv \lambda_1 n G(V) + \lambda_0 \left(1 - n\right)$
(iii) free entry of firms: determine $p(.)$;	(iii) free entry of firms: relate O and G;
(iv) stationarity: PDE of G	(iv) stationarity: PDE of G

5. Equilibrium

Develop a mapping ψ: $\quad w\,(V) \rightsquigarrow \psi w\,(V)$

- (a) start with a wage function $w(.)$
- (b) efficient sharing of values $\left[J'(V) = -1/u'(w)\right] \Longrightarrow$

$$J_w(V) = k/\bar{q} + \int_V^{\bar{V}} \frac{1}{u'(w(z))} dz$$

- (c) zero expected profit of recruiting:

$$q_w\,(V) = \frac{k}{J_w\,(V)}, \quad p_w(V) = M\left(\frac{k}{J_w(V)}\right)$$

Develop a mapping ψ (continued):

- (d) optimal application $\Longrightarrow F_w(V)$ and $S_w(V)$

- (e) Bellman equations for (J, V) and $\dot{J} = -\dot{V}/u'(w) \Longrightarrow$

$$\begin{aligned}
&\psi w(V) \\
=\ & y - [r + \lambda_1 p_w(F_w(V))] J_w(V) \\
&- \frac{1}{u'(w(V))} \max\{0, rV - \lambda_1 S_w(V) - u(w(V))\}
\end{aligned}$$

Determine the fixed point of ψ:

- assumption:
  - employment is worthwhile: $(0 <)\ b < \bar{w} = y - rk/\bar{q}$
  - lower bound on b to ensure: $\psi w \left(\underline{V}\right) \geq \underline{w}$
  - for all $w \in [\underline{w}, \bar{w}]$,

$$1 + \frac{u''(w)}{[u'(w)]^2}\left[u(\bar{w}) - u(w)\right] \geq 0$$

Determine the fixed point of ψ (continued):

- look for $w(V)$ in the following sets:

$$\begin{aligned}\Omega = \{w : \; & w(V) \in [\underline{w}, \bar{w}] \text{ for all } V; \\ & w(\bar{V}) = \bar{w}; \\ & w \text{ is continuous and (weakly) increasing}\}\end{aligned}$$

  Ω is closed and convex.

- for the equilibrium, we need $w(.)$ to lie in

$$\Omega' = \left\{w \in \Omega : w(V) \text{ is } \underline{\text{strictly}} \text{ increasing for all } V < \bar{V}\right\}$$

Theorem (Existence):

$$\psi \text{ has a fixed point } w^* \in \Omega'.$$

Proof (Schauder's fixed point theorem):

- Ω : nonempty, closed, bounded, and convex;
- $\psi : \Omega \to \Omega'$;
- ψ is continuous in sup norm;
- $\{\psi^j w\}_{j=0}^{\infty}$ is equi-continuous.

Properties of equilibrium (recap):

- wage-tenure relationship: $w'(V) > 0$

- wage-quit relationship: $\frac{d}{dV}\left[p\left(F\left(V\right)\right)\right] < 0$

- $p(V)$: strictly decreasing and strictly concave

$\Longrightarrow$ $F\left(V\right)$ is unique, and $F'\left(V\right) > 0$
(endogenous separation among applicants
and endogenously limited mobility)

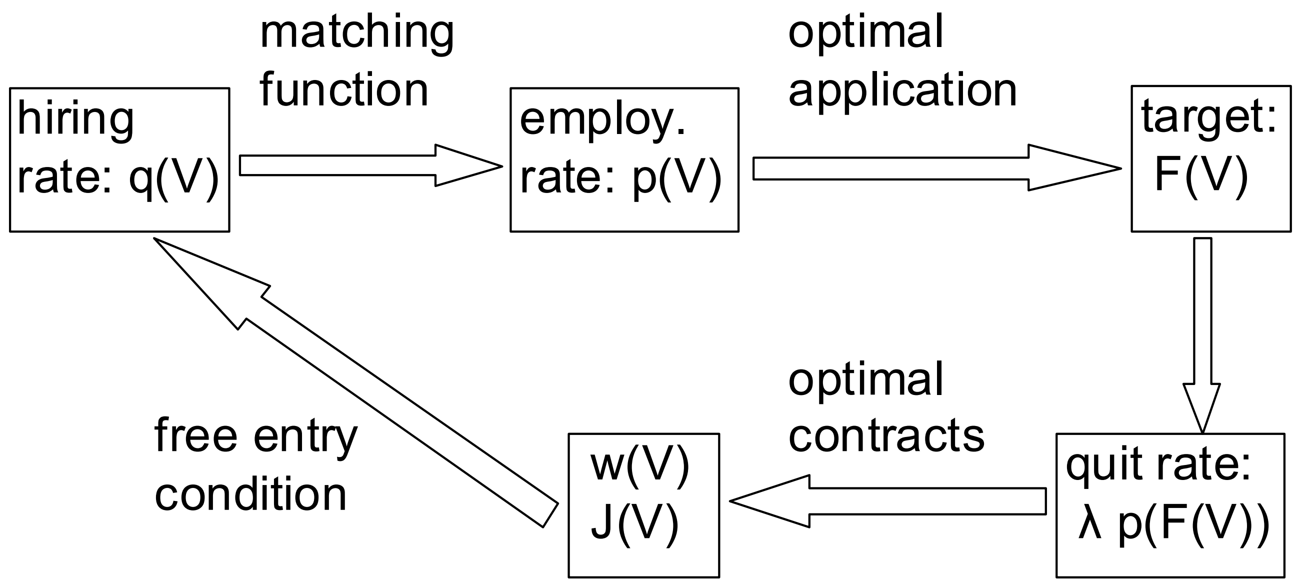
6. Comparative Statics

An increase in unemp. benefits, minimum wage or λ_0

- has **NO effect** on individual decisions such as
  - employed workers' optimal applications
  - equilibrium contracts
  - job-to-job transitions conditional on current wages

- it affects aggregate flows: through v_1 and distribution

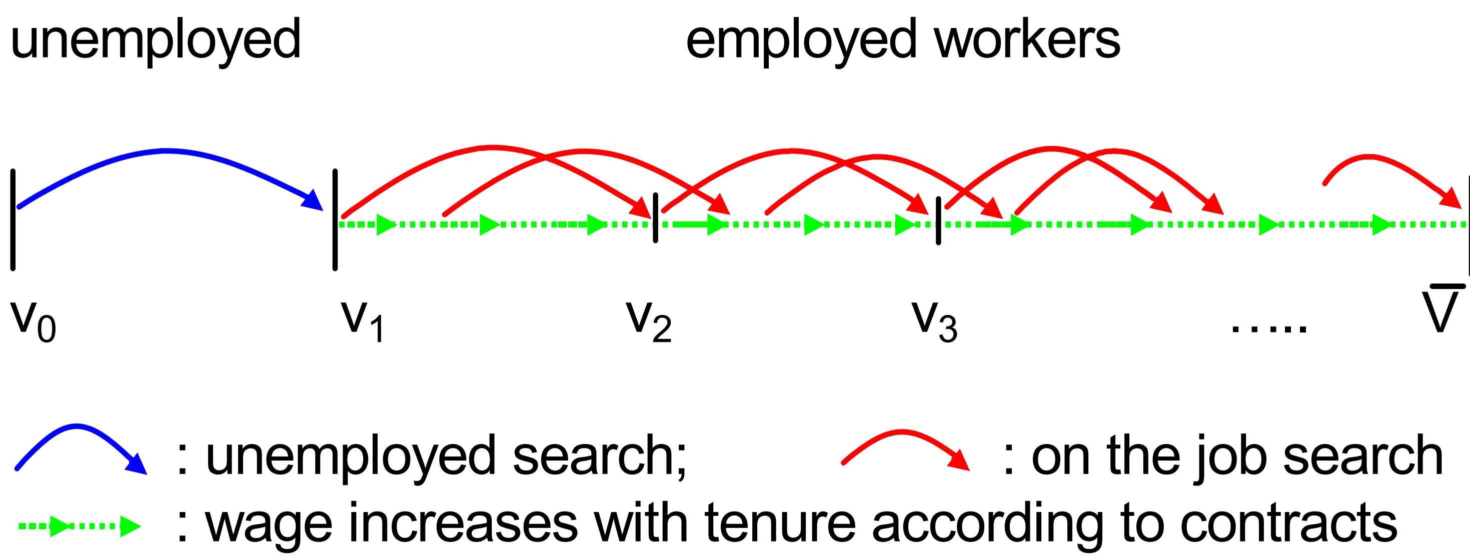
Why is there such independence?



Fixed-point problem

7. Nondegenerate Distribution of Workers:

(despite homogeneity and directed search)



$$v_0 = V_u, \quad v_j = F^{(j)}(v_0),\, j = 1, 2, ...$$

- CDF, $G(V)$, is continuous for all V:
  (consider any small time interval dt)
  - all existing workers at V move out:
    quit, or V increases over tenure
  - but inflow is proportional to dt
  - if there is a mass point at V, then
    outflow $>>$ inflow: a contradiction

- density g is continuously differentiable except at $v_2 = F(v_1)$

Theorem:

Equilibrium density function of employed workers is given by:

$$\frac{1}{\delta}g_1(V)\dot{V} = \Gamma(V, v_1) \equiv \exp\left[-\int_{T(v_1)}^{T(V)} \left[\delta + p\left(F\left(V\left(t\right)\right)\right)\right] dt\right]$$

$$g_j(V)\dot{V} - g_j(v_j)\dot{v}_j\Gamma(V, v_j)$$

$$= \lambda_1 \int_{v_j}^{V} \Gamma(V, z)p(z)g_{j-1}(F^{-1}(z))dF^{-1}(z), \quad j = 2, 3, ...$$

Moreover, $g_j(v_j) = \lim_{V \to v_j} g_{j-1}(V)$ for all j.

Density of employed workers:

- $\mathbf{g}(\mathbf{v}_1) = \mathbf{0}$ and $g'(v_1) > 0$

- if $F'(\bar{V}) > 0$, then $\mathbf{g}(\bar{\mathbf{V}}) = \mathbf{0}$:
so the density is decreasing for V close to $\bar{V}$

- density of wages is decreasing for w close to $\bar{w}$

Explain why $g(\bar{V}) = 0$:

(consider V such that $F(V) < \bar{V}$)

To support $F(V)$ as optimal application:

- $p(V') < p(F(V))$ for all $V' > F(V)$
- in particular, $p(\bar{V}) = 0$
- few firms recruit at values close to $\bar{V}$
- few workers are employed at these values.

What if search is undirected?

- an offer **does not** affect applications
  $\Longrightarrow$ tight connection between distribution and matching

- $\lambda_1 n G(V) + \lambda_0 (1 - n) = q(V) = \frac{k}{J(V)}$

- decreasing and concave $J(V)$
  $\Longrightarrow$ $g(V)$ is increasing (counterfactual!)

A computed example:

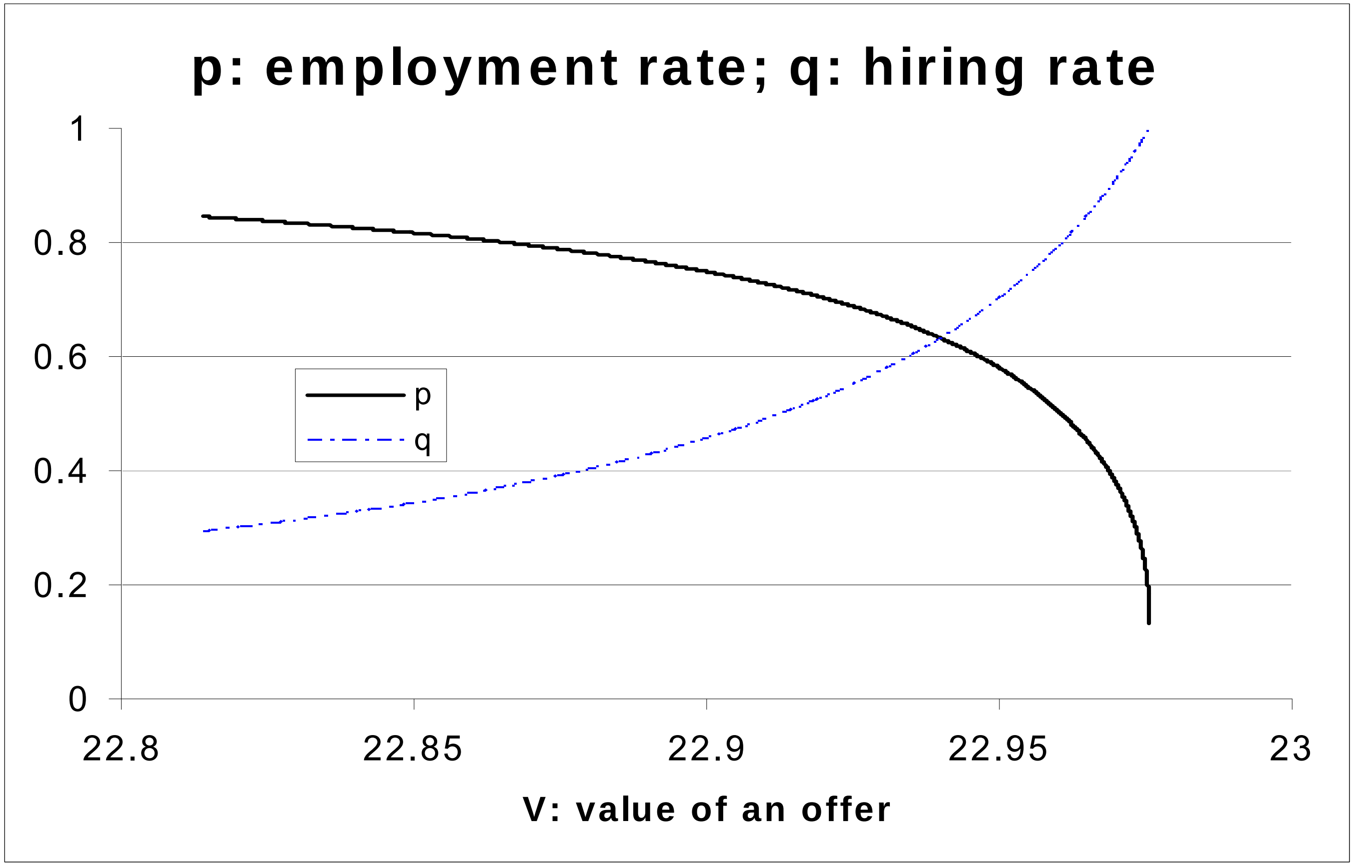
- utility function: $u(w) = \frac{w^{1-\eta}-1}{1-\eta}$
- urn-ball matching function:

$$\mathcal{M}(\theta, 1) = \bar{q}\left(1 - e^{-\theta}\right) \implies M(q) = -\frac{q}{\ln(1 - q/\bar{q})}$$

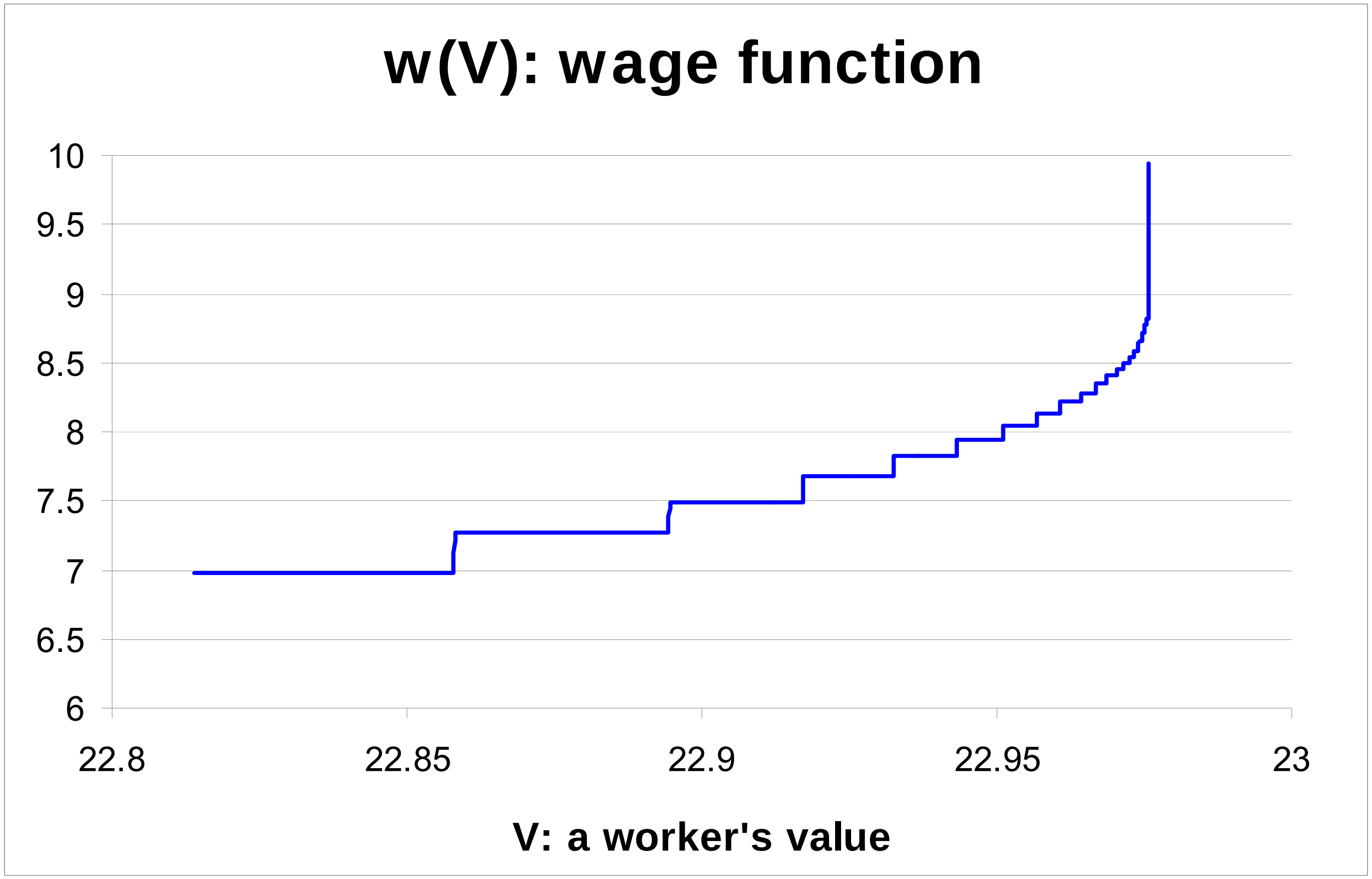
- parameter values:

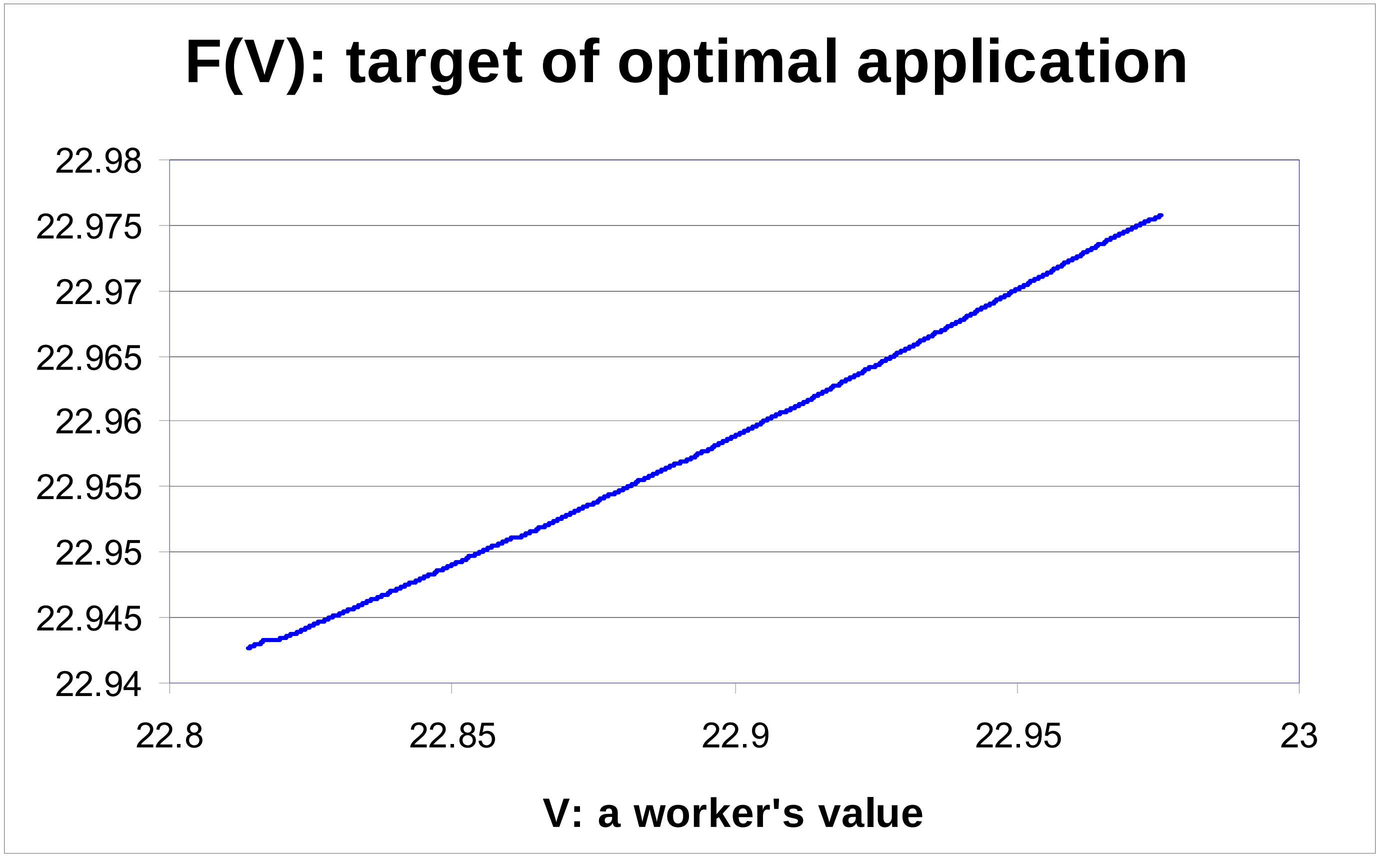
$$\eta = 1, \quad \bar{q} = 1, \quad y = 10, \quad k = 0.5$$

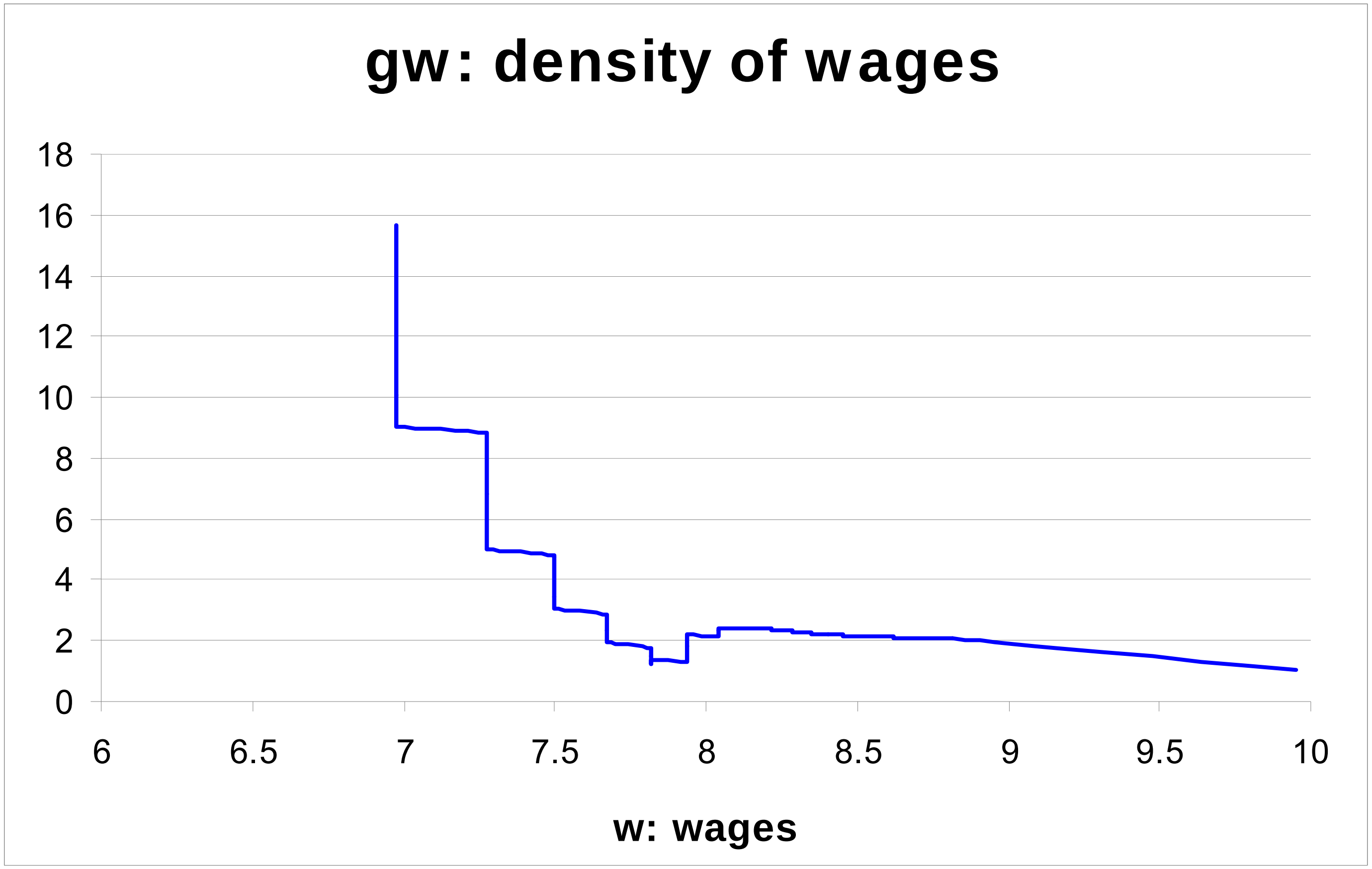
$$\delta = 0.1, \quad \rho = 0, \quad b = 2, \quad \lambda_0 = 1 = \lambda_1$$

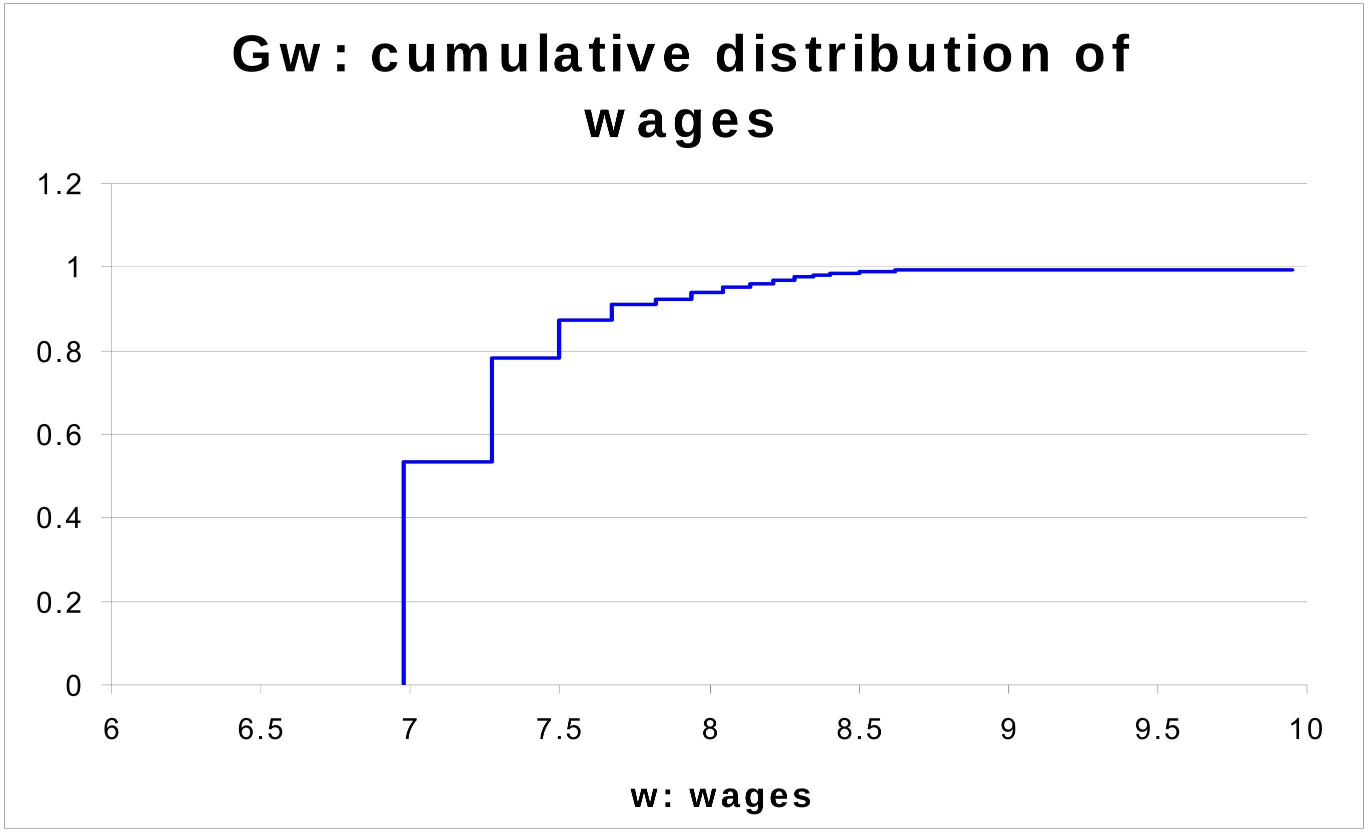


p: employment rate; q: hiring rate
1
0.8
0.6
0.4
0.2
0
22.8
22.85
22.9
22.95
23
V: value of an offer
p
q


w(V): wage function
10
9.5
9
8.5
8
7.5
7
6.5
6
22.8
22.85
22.9
22.95
23
V: a worker's value


F(V): target of optimal application
22.98
22.975
22.97
22.965
22.96
22.955
22.95
22.945
22.94
22.8
22.85
22.9
22.95
23
V: a worker's value


gw: density of wages
18
16
14
12
10
8
6
4
2
0
6
6.5
7
7.5
8
8.5
9
9.5
10
w: wages


Gw: cumulative distribution of wages
1.2
1
0.8
0.6
0.4
0.2
0
6
6.5
7
7.5
8
8.5
9
9.5
10
w: wages

8. Conclusion

- Directed OJS for wage contracts preserves:
  wage-tenure, wage-quit relationships
- New features:
  - block recursivity and tractability:
    individual decisions, contracts and matching rate functions are all independent of the distribution of workers
  - endogenously limited wage mobility and robust residual wage dispersion: exist even if all workers see all offers
  - wage density can be decreasing
  - new comparative statics results regarding policy