# Macroeconomic Theory

Katsuya Takii

# Introduction

- **Purpose:** This lecture is aimed at providing students with standard methods in modern macroeconomics. In particular, the lecture extends Solow model, which was taught in Macroeconomic Analysis, into three directions. Firstly, I apply the dynamic optimization techniques to endogenize saving rate in Solow model. Secondly, I introduce stochastic shocks to analyze uncertainty in a dynamic context. Finally, I discuss how one can numerically analyze the model. In addition, I also discuss how one can analyze discrete choices in a dynamic context. These methods are useful not only for understanding macroeconomics, but also for understanding dynamic issues in any fields of economics, such as public economics and financial economics.

# Introduction

- **Office hour:** At office, 17:00-18:00 on Monday. Appointment by e-mail is required for other time.
  - **Office:** 602 in OSIPP Building
  - **E-mail Address:** takii@osipp.osaka-u.ac.jp

- **Grading Policy:** 35% on assignments and 65% on a final exam.
  1. I will give you X assignments. Students must hand them in at the following lecture. If students turn an assignment in by the due date, I will give them 35/X points. If students turn an assignment in late, I will give them 14/X points. If students submit all assignments, you will receive 35 points. Students must write their answers with a pen. I don't allow the typed answers for this assignment.
  2. The full score of final exam is 65 points. I guarantee that exam questions with a score of 40 or higher out of 65 will be similar to those on assignment and lecture slides. (Let us think how to conduct the final exam.)

- **Remarks:**
  1. I assume that students have already taken the same level courses as Microeconomic Theory and Macroeconomics Analysis.
  2. Because of the nature of the issues, the lectures are rather technical. Students are expected to prepare by themselves to understand the lecture. I highly recommend this course to the students who think of economics as a major discipline and go to the doctoral program.
  3. I will teach this course in English unless all students prefer Japanese. I encourage students to make comments and questions in English. However, I will not prevent students from asking questions in Japanese. I can discuss your questions and comments in Japanese or English at my office hour.

# Introduction

- **Course Outline**
  1. *Basic Dynamic Programming (6 lectures)*
     - *Consumption, Optimal Growth Model and Recursive Competitive Equilibrium*
     - *Appendix: Overlapping Generation Model*
  2. *Continuous Time Dynamic Programing and Hamiltonian ( 2 lectures)*
     - *Investment and Continuous Optimal Growth Model*
  3. *Stochastic Dynamic Programming (3 lectures)*
     - *Asset Pricing and Stochastic Optimal Growth Model (Real Business Cycle Model)*
  4. *Dynamic Programing and Discrete Choice ( 3 lectures)*
     - *Labor Search and Equilibrium Unemployment Model*
  5. *Final Exam (1 lecture)*

# Introduction to Dynamic Programming

- Macroeconomics is a study to explain the behavior of aggregate data such as GDP per capita, inflation rate, and unemployment rate.
- For this purpose, we must infer the structure of the economy that brings the observable data.
- In order to infer the structure, we need a theory. There are two current consensuses among macroeconomists
  1. Use the variants of dynamic stochastic general equilibrium models to analyze economy.
  2. Developing a quantitative theory is useful.
     - Theory can be used not only for conveying the idea, but also for extracting meaningful information from data.
- The foundation of current macroeconomics that brings two consensuses is the neoclassical growth model. This is the main subject of this lecture.

# Introduction to Dynamic Programming

- Remember Kaldor's Stylized Facts (1963)
  1. *The growth rate of GDP per capita is nearly constant.*
  2. *The growth rate of capital per capita is nearly constant.*
  3. *The growth rate of output per worker differs substantially across countries.*
  4. *The rate of return to capital is nearly constant.*
  5. *The ratio of physical capital to output is nearly constant.*
  6. *The shares of labor and physical capital are nearly constant.*
- In order to meet this requirement, the growth model must exhibit the balanced growth path.

- Our candidate model is the extended Solow Model, which is summarized by the following 3 equations.

$$
\begin{aligned}
K_{t+1} &= F\left(K_t, T_t N_t\right) + (1-\delta) K_t - C_t \\
T_{t+1} &= (1+g) T_t \\
N_{t+1} &= (1+n) N_t
\end{aligned}
$$

with an assumption.

$$C_t = (1-s) F\left(K_t, T_t N_t\right)$$

# Lucas's Critique and Micro Foundation

- The saving rate $s$ is the results of individual behaviors. Changes in environment may influence the properties of these behaviors.
- Lucas (1976) argues that as people make decisions based on their expectation on the future economic environment, the expected future policy change is likely to influence their expectation, and, therefore, their decisions. It means that the estimated parameters on consumption functions and, therefore, the saving rate is likely to change when a government changes its policy. So we cannot use the estimated parameters for policy simulations.
- Following Lucas's critique, many macroeconomists pay more attention to the micro foundation of the consumer's behavior and firm's behavior, and derives consumption function, investment function
- We would like to develop a model which allows us to analyze the impact of a policy change on real economy through changes in the saving rate.

- In order to endogenously determine the saving rate, we must know how consumers decide their consumption.
- One of a convenient assumption is that a representative consumer maximizes the following utility function.

$$\max_{\{c_t\}_0^\infty} \sum_{t=0}^{\infty} \beta^t N_t U(c_t),$$

$$\begin{aligned} K_{t+1} &= F(K_t, T_t N_t) + (1-\delta) K_t - c_t N_t, \ \ K_0 \ \textit{is given} \\ T_{t+1} &= (1+g) T_t, \ \ T_0 \ \textit{is given} \\ N_{t+1} &= (1+n) N_t, \ \ N_0 \ \textit{is given} \end{aligned}$$

where $c_t = \frac{C_t}{N_t}$.

- This model is called the neoclassical growth model. In order to solve this problem, we need to know how to solve a dynamic optimization problem.

# A Finite Horizon Problem

- Consider the following problem

$$\max_{\{X_t\}_{t=\tau}^{T-1}} \left\{ \sum_{t=\tau}^{T-1} \beta^{(t-\tau)} r(X_t, S_t) + \beta^{(T-\tau)} V_T(S_T) \right\},$$

$$\begin{aligned} s.t.\ S_{t+1} &= G(X_t, S_t), \\ & S_\tau \text{ is given.} \end{aligned}$$

- The variable, $\{X_t\}$, is called a control variable. It can be a vector. This is the variable which an agent tries to control in order to maximize his objective function. The variable $\{c_t\}$ is the control variable in the neoclassical growth model.
- The variable, $\{S_t\}$, is called a state variable. It can be a vector. The state variable summarizes the current state of the economy. The vector $\{(K_t, T_t, N_t)\}$ is a state vector in the neoclassical growth model.

# A Finite Horizon Problem

- The function $r(X_t, S_t)$ is called the one period return function. One period return function summarizes the instantaneous reward from current state variable and the current control variable. In the case of the neoclassical growth model, $r(X_t, S_t) = N_t U(c_t)$.
- The function $G(X_t, S_t)$ is called the transition function. It describes the dynamic behavior of the state variables. In the case of the neoclassical growth model

$$G(c_t, (K_t, T_t, N_t)) = \left\{ \begin{array}{c} F(K_t, T_t N_t) + (1-\delta) K_t - c_t N_t \\ (1+g) T_t \\ (1+n) N_t \end{array} \right\}$$

- The function, $V_T(\cdot)$ is called the value function at the last period. Since the state variable summarizes the current state of economy. The value function indicates, how much value he expects to obtain in the future when the current state is $S_T$.

# A Finite Horizon Problem

- Given $S_\tau$, when the agent chooses $X_\tau$ at date $\tau$, $S_{\tau+1}$ is automatically determined. Given this $S_{\tau+1}$, the agent can choose $X_{\tau+1}$ at date $\tau+1$, and it determines $S_{\tau+2}$, and so on. In this way, the agent can recursively decide his decisions.
- Note that

$$\begin{aligned}
& \max_{\{X_t\}} \left\{ \sum_{t=\tau}^{T-1} \beta^{(t-\tau)} r\left(X_t, S_t\right) + \beta^{(T-\tau)} V_T\left(S_T\right) \right\} \\
= & \max_{\{X_t\}} \left\{ \begin{array}{c} .... + \beta^{(t-1-\tau)} r\left(X_{t-1}, S_{t-1}\right) + \beta^{(t-\tau)} r\left(X_t, S_t\right) \\ + \beta^{(t+1-\tau)} r\left(X_{t+1}, S_{t+1}\right) + ... \beta^{(T-\tau)} V_T\left(S_T\right) \end{array} \right\} \\
= & \max_{\{X_s\}_{s<t}} \left\{ \begin{array}{c} .... + \beta^{(t-1-\tau)} r\left(X_{t-1}, S_{t-1}\right) \\ + \max_{\{X_x\}_{x\geq t}} \left[ \begin{array}{c} \beta^{(t-\tau)} r\left(X_t, S_t\right) \\ + \beta^{(t+1-\tau)} r\left(X_{t+1}, S_{t+1}\right) + \\ ... \beta^{(T-\tau)} V_T\left(S_T\right) \end{array} \right] \end{array} \right\}
\end{aligned}$$

# A Finite Horizon Problem

- That is, once the agent knows, $S_t$, he does not need to worry about other past variables $\{(X_s, S_s)\}_{s<t}$ because the state variable summarizes every important information of the economy at that time.

- Hence, the maximization problem at date $T-1$ can be simplified as follows:

$$\max_{X_{T-1}} \{r(X_{T-1}, S_{T-1}) + \beta V_T(G(X_{T-1}, S_{T-1}))\},$$
$$S_{T-1} \textit{ is given.}$$

- The solution to this problem depends on $S_{T-1}$. That is, we can derive an optimal policy function $x_{T-1}(\cdot)$ as a function of $S_{T-1}$:

$$x_{T-1}(S_{T-1}) \equiv \arg\max_{X_{T-1}} \{r(X_{T-1}, S_{T-1}) + \beta V_T(G(X_{T-1}, S_{T-1}))\}$$

# A Finite Horizon Problem

- Since the policy function is an optimal strategy, we can define the value function at date $T-1$, $V_{T-1}(\cdot)$, as follows:

$$\begin{aligned} & \max_{X_{T-1}} \left\{ r\left(X_{T-1}, S_{T-1}\right) + \beta V_T\left(G\left(X_{T-1}, S_{T-1}\right)\right) \right\} \\ = \ & r\left(x_{T-1}\left(S_{T-1}\right), S_{T-1}\right) + \beta V_T\left(G\left(x_{T-1}\left(S_{T-1}\right), S_{T-1}\right)\right) \\ \equiv \ & V_{T-1}\left(S_{T-1}\right). \end{aligned}$$

- Using $V_{T-1}(S_{T-1})$ we can rewrite the agent's problem at dates $T-2$ as follows:

$$\begin{aligned} & \max_{X_{T-2}} \left\{ r\left(X_{T-2}, S_{T-2}\right) + \beta V_{T-1}\left(G\left(X_{T-2}, S_{T-2}\right)\right) \right\}, \\ & S_{T-2} \ \textit{is given}. \end{aligned}$$

- This is the similar problem as before. Therefore, we can define $V_{T-2}(S_{T-2})$ and using $V_{T-2}(S_{T-2})$, we can describe the problem at date $T-3$ and so on.

# A Finite Horizon Problem

- In general, an original problem can be rewritten as the sequence of a simple static problem:

$$V_t(S_t) \equiv \max_{X_t} \{r(X_t, S_t) + \beta V_{t+1}(G(X_t, S_t))\},$$

  This equation is called the Bellman Equation.
- Hence, a finite horizon problem can be solved by a sequence of policy functions and value functions, $\{(x_t(\cdot),\ V_t(\cdot))\}_{t=0}^{T-1}$.
- For any initial state variable, $S_\tau$, the optimal policy function $x_\tau(S_\tau)$ determines $X_\tau$ and the transition function $G(X_\tau, S_\tau)$ determines $S_{\tau+1}$. Given this $S_{\tau+1}$, the optimal policy function and the transition function determines $X_{\tau+1}$ and $S_{\tau+2}$ and so on.

# A Finite Horizon Problem

- Assume that the first order condition is valid. The first order condition is

$$0 = r_1\left(x_t\left(S_t\right), S_t\right) + \beta V'_{t+1}\left(G\left(x_t\left(S_t\right), S_t\right)\right) G_1\left(x_t\left(S_t\right), S_t\right), \quad (1)$$

  for any $t$.
- The first term of the right hand side of equation (1) is the marginal return from changing $X_t$. However, when the agent changes $X_t$, it affects not only the current return, but also the future returns by changing the future state variable. When the agent slightly changes $X_t$, it will change $S_{t+1}$ by $G_1\left(X_t, S_t\right)$. If $S_{t+1}$slightly moves, it changes the present value of the future reward at date $t+1$ by $V'_{t+1}\left(S_{t+1}\right)$. Since the agent discount the future by $\beta$, the future impact of changing $X_t$ is $\beta V'_{t+1}\left(G\left(X_t, S_t\right)\right) G_1\left(X_t, S_t\right)$. If the agent optimally chooses variables, these two effects must be the same at any date $t$.

# A Finite Horizon Problem

- It is useful to derive the envelope theorem:

$$
\begin{aligned}
& V_t'(S_t) \\
= \; & r_2(x_t(S_t), S_t) + \beta V_{t+1}'(G(x_t(S_t), S_t)) G_2(x_t(S_t), S_t) + \\
& \left[ r_1(x_t(S_t), S_t) + \beta V_{t+1}'(G(x_t(S_t), S_t)) G_1(x_t(S_t), S_t) \right] x_t'(S_t) \\
= \; & r_2(x_t(S_t), S_t) + \beta V_{t+1}'(G(x_t(S_t), S_t)) G_2(x_t(S_t), S_t)
\end{aligned}
$$

for any $t$.

# A Finite Horizon Problem

- It shows the marginal benefit of changing the state variable, $S_t$. When the state variable changes a little bit, it changes the current return by $r_2\left(x_t\left(S_t\right), S_t\right)$. But since changing the state variable at date $t$ will change the future state variable $S_{t+1}$ by $G_2\left(X_t, S_t\right)$, it also has a dynamic effect. Since changing $S_{t+1}$ affects the future reword by $V_{t+1}'\left(S_{t+1}\right)$ and the agent discounts the future by $\beta$, the total effect must be $\beta V_{t+1}'\left(G\left(X_t, S_t\right)\right) G_2\left(X_t, S_t\right)$.

# A Finite Horizon Problem

- *Example:* Consider the following household problem,

$$\max_{\{c_t\}} \sum_{t=\tau}^{T-1} \beta^{(t-\tau)} U(c_t) + \beta^{(T-\tau)} V_T(a_T),$$
$$s.t.\ a_{t+1} = (1+\rho) a_t + w - c_t$$
$$given\ a_\tau$$

- We can define the Bellman equation as follows:

$$V_t(a_t) = \max_{c_t} \{U(c_t) + \beta V_{t+1}(a_{t+1})\},$$
$$s.t.\ a_{t+1} = (1+\rho) a_t + w - c_t,$$

# A Finite Horizon Problem

- The first order condition is

$$U'\left(c\left(a_t\right)\right) = \beta V'_{t+1}\left(\left(1+\rho\right)a_t + w - c\left(a_t\right)\right)$$

- Envelope theorem

$$V'_t\left(a_t\right) = \beta V'_{t+1}\left[\left(1+\rho\right)a_t + w - c\left(a_t\right)\right]\left[1+\rho\right].$$

- Combining Two equations

$$\frac{U'\left(c\left(a_{t-1}\right)\right)}{\beta} = U'\left(c\left(a_t\right)\right)\left(1+\rho\right)$$

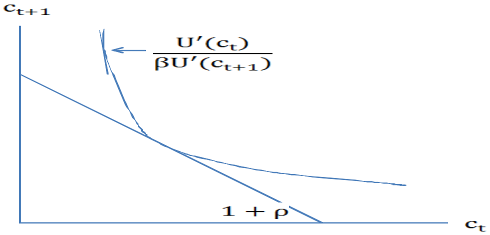
# A Finite Horizon Problem

- Euler equation:

$$\frac{U'(c_t)}{\beta U'(c_{t+1})} = 1 + \rho$$

where $c_{t+1} = c(a_{t+1})$ and $c_t = c(a_t)$. The left hand side is the marginal rate of substitution between consumption at date $t$ and $t+1$; the right hand side is the marginal gain from saving.

# Euler Equation



# An Infinite Horizon Problem

- Now let me consider the case, $T$ goes infinite. That is, an original problem is

$$\begin{aligned} U(S_\tau, \tau) &= \max_{\{X_t\}} \left\{ \sum_{t=\tau}^{\infty} \beta^{(t-\tau)} r(X_t, S_t) \right\}, \\ s.t.\ S_{t+1} &= G(X_t, S_t), \ \ S_\tau \ is\ given. \end{aligned}$$

- If $T$ goes infinite, there is no last period and we cannot use the previous method. However, we can explain it by analogy between this model and the previous one.
- Assume that a policy function of this original problem is $x_u(S_t)$.

# An Infinite Horizon Problem

- Remember that once we control a current state, the past decision does not influence future maximization problem. Hence,

$$
\begin{aligned}
U(S_\tau, \tau) &= \max_{\{X_t\}_\tau^\infty} \sum_{t=\tau}^{\infty} \beta^{(t-\tau)} r(X_t, S_t), \\
&= \max_{X_\tau} \left\{ r(X_\tau, S_\tau) + \beta \max_{\{X_t\}_{\tau+1}^\infty} \sum_{t=\tau+1}^{\infty} \beta^{(t-(\tau+1))} r(X_t, S_t) \right\}, \\
&= \max_{X_\tau} \left\{ r(X_\tau, S_\tau) + \beta U(S_{\tau+1}, \tau+1) \right\}
\end{aligned}
$$

# An Infinite Horizon Problem

- Hence, we can consider the following recursive problem:

$$\begin{aligned} V(S_t) &= \max_{X_t} \{ r(X_t, S_t) + \beta V(S_{t+1}) \}, \\ S_{t+1} &= G(X_t, S_t). \end{aligned} \tag{2}$$

- Assume that the policy function of this problem is $x(S_t)$.
  1. With certain mild conditions ($r(X, S)$ is bounded and continuous, $\{(S_{t+1}, X_t, S_t) \mid S_{t+1} \leq G(X_t, S_t)\}$ is compact), we can show that there exists a unique value function $V(\cdot)$ and a policy function which solves equation (2) and that $U(S_\tau, \tau) = V(S_\tau)$ and $x_u(S_\tau) = x(S_\tau)$. That is, Bellman equation (2) is equivalent to the original problem and the value function and policy function is time invariant.
  2. With the further assumptions ($r(X, S)$ is strict concave and $\{(S_{t+1}, X_t, S_t) \mid S_{t+1} \leq G(X_t, S_t)\}$ is convex), $V(S)$ is strictly concave and $x(S_\tau)$ is continuous and single valued.
  3. With further mild condition ( $r(X, S)$ is continuously differentiable), it is known that $V$ is continuously differentiable.

# An Infinite Horizon Problem

- An economic interpretation of the Bellman equation (2) is that the agent maximizes the sum of current return, $r(X_t, S_t)$, and the discounted future values from $S_{t+1}$, $\beta V(S_{t+1})$. He is concerned about the trade-off between the current benefits and the future benefits when he chooses $X_t$. Since he lives forever, it does not matter when he makes his decisions. Therefore, the value function and the policy function do not depend on time. His problem is stationary.

# An Infinite Horizon Problem

- There are several methods to analyze the Bellman equation (2):
  1. Guess and Verify Method
  2. Euler Equation
  3. Numerical method.

# Numerical Method

- The original problem can be approximated by a finite horizon problem:

$$
\begin{aligned}
U(S_\tau,\tau) &= \lim_{T\to\infty}\max_{\{X_t\}}\left\{\left[\sum_{t=\tau}^{T-1}\beta^{(t-\tau)}r(X_t,S_t)+\beta^{(T-\tau)}V_T(S_T)\right]\right\}, \\
s.t.\ S_{t+1} &= G(X_t,S_t), \\
& S_\tau \text{ is given.}
\end{aligned}
$$

- It is shown that we can approximate the Bellman equation (2) by the corresponding finite horizon problem.

$$
\begin{aligned}
V(S) &= \lim_{t\to-\infty} V_t(S). \\
V_t(S) &\equiv \max_X\{r(X,S)+\beta V_{t+1}(G(X,S))\} \qquad (3)
\end{aligned}
$$

Since there exist a unique value function $V(\cdot)$, if we iterate equation (3) from an arbitrary initial value function $V_T(S)$ for any $S$, it must converge to $V(S)$.

# An Infinite Horizon Problem

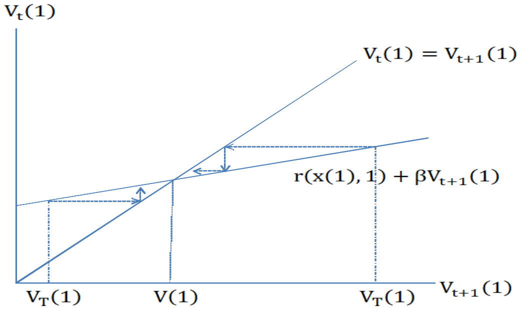
- **Example:** Suppose that $S \in \{1\}$. Therefore, $S_{t+1} = G(X_t, S_t) = 1$ for any $t$. Suppose that.

$$\begin{aligned} x_t(1) &= \arg\max_X \{r(X,1) + \beta V_{t+1}(G(X,1))\} \\ &= \arg\max_X \{r(X,1) + \beta V_{t+1}(1)\} \\ &= \arg\max_X \{r(X,1)\} \equiv x(1) \end{aligned}$$

Then

$$V_t(1) \equiv r(x(1),1) + \beta V_{t+1}(1).$$

## The Existence of a Globally Stable Unique Value Function



- **Guess and Verify method:** One way to analyze the property of equation (2) is guess and very method. The first, guess what would be the property of $V(\cdot)$ and assume that the property is true. Then verify that your guess satisfy equation (2). Since we know the value function is unique, if a property satisfies equation (2), the value function must have the property.

# Guess and Verify Method

- *Example:* Consider the following problem.

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t),$$
$$s.t.\ a_{t+1} = (1+\rho)\, a_t + w - c_t$$

- Bellman Equation

$$V(a_t) = \max_{c_t} \{u(c_t) + \beta V((1+\rho)\, a_t + w - c_t)\}$$

# Guess and Verify Method

- Guess $V(a') \geq V(a)$ if $a' \geq a$. Suppose that

$$c(a_t) = \arg\max_{c_t} \{u(c_t) + \beta V((1+\rho) a_t + w - c_t)\}$$

Note that

$$\begin{aligned} & u(c(a'_t)) + \beta V((1+\rho) a'_t + w - c(a')) \\ \geq \quad & u(c(a_t)) + \beta V((1+\rho) a'_t + w - c(a_t)) \\ \geq \quad & u(c(a_t)) + \beta V((1+\rho) a_t + w - c(a_t)) \end{aligned}$$

Hence, our guess, $V(a') \geq V(a)$, is verified.

# Guess and Verify Method

- Since many applied economists restrict their attention to the case to which we can apply the first order condition, I explain this method by using the first order condition. The policy function $x(\cdot)$ must satisfy the first order condition and the bellman equation for any $S$ as follows:

$$0 = r_1(x(S), S) + \beta V'(G(x(S), S)) G_1(x(S), S) \quad (4)$$

$$V(S) = r(x(S), S) + \beta V(G(x(S), S)). \quad (5)$$

  Given the value function $V(\cdot)$, equation (4) determines $x(\cdot)$; given $x(\cdot)$, equation (5) determines $V(\cdot)$. That is, these are simultaneous equations. We can analyze our model by examining these two equations.
- In general we cannot solve a closed form solution. However, there is a special case in which we can find a closed form solution.

# Guess and Verify Method

- *Example:* Consider the following problem.

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t \ln c_t,$$
$$s.t.\ a_{t+1} = (1+\rho) a_t - c_t$$

- We can define the Bellman equation as follows:

$$V(a_t) = \max_{c_t} \{\ln c_t + \beta V(a_{t+1})\},$$
$$s.t.\ a_{t+1} = (1+\rho) a_t - c_t$$

# Guess and Verify Method

- Guess $V(a_t) = f + g \ln(a_t)$. Then consider the problem

$$\max_{c_t} \{\ln c_t + \beta [f + g \ln(a_{t+1})]\},$$
$$s.t. \; a_{t+1} = (1+\rho) a_t - c_t,$$

- First Order Condition implies

$$\frac{1}{c(a_t)} = \frac{\beta g}{(1+\rho) a_t - c(a_t)}$$

- Policy Function

$$c(a_t) = \frac{(1+\rho) a_t - c(a_t)}{\beta g}$$
$$\frac{\beta g + 1}{\beta g} c(a_t) = \frac{(1+\rho) a_t}{\beta g}$$
$$c(a_t) = \frac{(1+\rho) a_t}{\beta g + 1}$$

# Guess and Verify Method

- $a_{t+1}$

$$
\begin{aligned}
a_{t+1} &= (1+\rho)\,a_t - c\,(a_t) \\
&= (1+\rho)\,a_t - \frac{(1+\rho)\,a_t}{\beta g + 1} \\
&= \frac{\beta g\,(1+\rho)\,a_t}{\beta g + 1}
\end{aligned}
$$

# Guess and Verify Method

- The Maximized Value

$$\begin{aligned} & \ln c\left(a_t\right)+\beta\left[f+g\ln\left(a_{t+1}\right)\right] \\ = & \ln\frac{(1+\rho)\,a_t}{\beta g+1}+\beta\left[f+g\ln\left(\frac{\beta g\,(1+\rho)\,a_t}{\beta g+1}\right)\right] \\ = & (1+\beta g)\ln\left[\frac{(1+\rho)\,a_t}{\beta g+1}\right]+\beta\left[f+g\ln\beta g\right] \\ = & (1+\beta g)\ln\frac{(1+\rho)}{\beta g+1}+\beta\left[f+g\ln\beta g\right]+(1+\beta g)\ln a_t \end{aligned}$$

- Hence, if we can find $f$ and $g$ that satisfy

$$\begin{aligned} f &= (1+\beta g)\ln\frac{(1+\rho)}{\beta g+1}+\beta\left[f+g\ln\beta g\right] \\ g &= 1+\beta g \end{aligned}$$

then we can verify our guess.

# Guess and Verify Method

- $g$

$$g = \frac{1}{1-\beta}$$

- $f$

$$\begin{aligned}
f &= (1+\beta g)\ln\frac{(1+\rho)}{\beta g+1} + \beta\left[f + g\ln\beta g\right] \\
&= \frac{1}{1-\beta}\left[(1+\beta g)\ln\frac{(1+\rho)}{\beta g+1} + \beta g\ln\beta g\right] \\
&= \frac{1}{1-\beta}\left[\left(1+\frac{\beta}{1-\beta}\right)\ln\frac{(1+\rho)}{\frac{\beta}{1-\beta}+1} + \frac{\beta}{1-\beta}\ln\frac{\beta}{1-\beta}\right] \\
&= \frac{1}{1-\beta}\left[\left(1+\frac{\beta}{1-\beta}\right)\ln(1+\rho)(1-\beta) + \frac{\beta}{1-\beta}\ln\frac{\beta}{1-\beta}\right] \\
&= \frac{1}{1-\beta}\left[\frac{\ln(1+\rho)}{1-\beta} + \frac{\beta}{1-\beta}\ln\beta + \ln(1-\beta)\right]
\end{aligned}$$

# Guess and Verify Method

- $V(a_t)$

$$V(a_t) = \frac{1}{1-\beta}\left[\frac{\ln(1+\rho)}{1-\beta} + \frac{\beta}{1-\beta}\ln\beta + \ln(1-\beta) + \ln a_t\right]$$

# Guess and Verify Method

- The guess and verify method demands many calculations. If we are only interested in a policy function, but not a value function, there is one way to reduce calculation: the use of envelope theorem.
- The first order condition and the envelope theorem of the Bellman equation (2) imply that

$$
\begin{aligned}
0 &= r_1(x(S), S) + \beta V'(G(x(S), S)) G_1(x(S), S) && (6)\\
V'(S) &= r_2(x(S), S) + \beta V'(G(x(S), S)) G_2(x(S), S) && (7)
\end{aligned}
$$

for any $S_t$. Again two unknown functions $x(\cdot)$ and $V'(\cdot)$ can be solved by two equations.

# Guess and Verify Method

- Note that the first order condition (6) is the same as the first order condition (4); the envelope theorem (7) is not the same as the Bellman equation (5).
- Also note that the previous two equations determine $x(\cdot)$ and $V(\cdot)$; the current two conditions determine $x(\cdot)$ and $V'(\cdot)$. In general, there exists $C$ such that

$$V(S) = \int V'(S)\, dS + C.$$

Since equation (6) and (7) cannot determine $C$, without a boundary condition, equation (6) and (7) cannot solve the value function. However, if we are only interested in a policy function, equations (6) and (7) can solve it.

# Guess and Verify Method

- *Example:* Consider the previous problem again.

$$V(a_t) = \max_{c_t} \{\ln c_t + \beta V(a_{t+1})\},$$
$$s.t. \quad a_{t+1} = (1+\rho)\, a_t - c_t$$

- The first order condition and envelope theorem are

$$\frac{1}{c(a_t)} = \beta V'((1+\rho)\, a_t - c(a_t))$$
$$V'(a_t) = \beta (1+\rho) V'((1+\rho)\, a_t - c(a_t)),$$

# Guess and Verify Method

- Guess $V'(a) = \frac{g}{a}$. Then the first order condition is

$$\frac{1}{c(a_t)} = \frac{g\beta}{(1+\rho)a_t - c(a_t)}$$

- Policy function

$$\begin{aligned} c(a_t) &= \frac{(1+\rho)a_t - c(a_t)}{\beta g} \\ \frac{\beta g + 1}{\beta g} c(a_t) &= \frac{(1+\rho)a_t}{\beta g} \\ c(a_t) &= \frac{(1+\rho)a_t}{\beta g + 1} \end{aligned}$$

# Guess and Verify Method

- Hence,

$$\begin{aligned}
& \beta (1+\rho) V' ((1+\rho) a_t - c(a_t)) \\
= & \frac{(1+\rho)}{c(a_t)} \\
= & \frac{(1+\rho)}{\frac{(1+\rho)a_t}{\beta g+1}} = \frac{\beta g+1}{a_t}
\end{aligned}$$

- Hence if we find $g$ that satisfies

$$g = \beta g + 1,$$

our guess is satisfied.

# Guess and Verify Method

- $g$

$$g = \frac{1}{1-\beta}.$$

- $V'(a)$

$$V'(a) = \frac{1}{(1-\beta)\,a}$$

# Guess and Verify Method

- **Assignment:** Suppose that $r(I_t, K_t) = \frac{1}{1+\rho}[rK_t - I_t - C(I_t)]$, $K_{t+1} = G(I_t, K_t) = I_t + (1-\delta)K_t$ and $\beta = \frac{1}{1+\rho}$ where $K_t$ is capital stock, $I_t$ is investment, $r$ is the rental price of capital, $\delta$ is the depreciation rate, $\rho$ is an real interest rate and $C(\cdot)$ is the adjustment cost of investment, $C'(I_t) > 0$ and $C''(I_t) > 0$. A firm chooses $I$ to maximizes the present value of the stream of $r(I_t, K_t)$.
  1. Define the Bellman Equation.
  2. Show that the value function is an affine in $K$: that is, there exist $q$ and $m$ such that $V(K) = qK + m$.
  3. Show that the investment $I$ is a function of $q$.
  4. Show that $q$ must satisfy
  $$q = \frac{r}{\rho + \delta}$$
  5. Suppose that $C(I) = \frac{A}{2}I^2$. Derive an investment function.

# Guess and Verify Method

**Assignment:** Suppose that $r(I_t, K_t) = \frac{1}{1+\rho}[rK_t - pI_t - C(I_t, K_t)]$, $K_{t+1} = G(I_t, K_t) = I_t + (1-\delta) K_t$ and $\beta = \frac{1}{1+\rho}$ where $K_t$ is capital stock, $I_t$ is investment, $r$ is the rental price of capital, $\delta$ is the depreciation rate, $p$ is the relative price of investment, $\rho$ is an real interest rate, and $C_I(I_t, K_t) > 0$ $C_{II}(I_t, K_t) > 0$ and $C_K(I_t, K_t) < 0$, $C(I_t, K_t)$ is the constant returns to scale in $I$ and $K$. A firm chooses $I$ to maximizes the present value of the stream of $r(I_t, K_t)$. Assume that there exists $B$ such that $0 < \frac{\frac{I_t}{K_t}+(1-\delta)}{1+\rho} \leq B < 1$.

- 
  1. Define the Bellman Equation.
  2. Show that the value function is linear in $K$: that is, there exists $q$ such that $V(K) = qK$.
  3. Show that the investment to capital stock, $\frac{I}{K}$, is a function of Tobin's $Q$: $q^{Tobin} = \frac{V(K)}{pK} = \frac{q}{p}$.

# Euler Equation

- **Euler Equation:** In general, it is difficult to solve a closed form solution. However, it is possible to analyze the property of solutions for more general class of functions. Remember that the first order condition and envelope theorem are

$$
\begin{aligned}
0 &= r_1(x(S), S) + \beta V'(G(x(S), S))\, G_1(x(S), S), \\
V'(S) &= r_2(x(S), S) + \beta V'(G(x(S), S))\, G_2(x(S), S).
\end{aligned}
$$

- Using two equations, we can eliminate $V'(S)$ and derive the first order nonlinear difference equation. This is called Euler equation.

$$
\frac{r_1(X_t, S_t)}{\beta G_1(X_t, S_t)} = \frac{r_1(X_{t+1}, S_{t+1})}{G_1(X_{t+1}, S_{t+1})} G_2(X_{t+1}, S_{t+1}) - r_2(X_{t+1}, S_{t+1}). \tag{8}
$$

# Euler Equation

- Although the first order condition and the envelope theorem can derive the Euler equation, the Euler equation cannot derive the first order condition and the envelope theorem. That is, the elimination of $V'(S_t)$ discards useful information. Therefore, the Euler equation is a necessary condition for the original problem, but not sufficient.
- Note that we can derive the dynamics of $X_t$ from Euler equation:

$$X_t = \Phi(S_{t+1}, X_{t+1}, S_t)$$

- We also have the transition equation for the dynamics of $S$.

$$S_{t+1} = G(X_t, S_t),$$

# Euler Equation

- It means that

$$
\begin{aligned}
X_\tau &= \Phi\left(S_{\tau+1}, X_{\tau+1}, S_\tau\right) \\
&.. \\
X_{T-1} &= \Phi\left(S_T, X_T, S_{T-1}\right) \\
&..
\end{aligned}
$$

$$
\begin{aligned}
S_{\tau+1} &= G\left(X_\tau, S_\tau\right) \\
&.. \\
S_T &= G\left(X_{T-1}, S_{T-1}\right) \\
&..
\end{aligned}
$$

# Euler Equation

- These equations imply that from period $\tau$ to period $T-1$, the number of equations are $2(T-\tau)$ and the number of $X_t$ and $S_t$ is $2(T-\tau+1)$. It means that we need 2 known variables to solve the sequence of $X_t$ and $S_t$ between $\tau$ and $T-1$.
- Now we have one initial condition $S_\tau$. In a finite horizon case, since the value function at the end period is given, the first order condition,

$$r_1(X_T, S_T) + \beta V'_{T+1}(G(X_T, S_T))\, G_1(X_T, S_T) = 0,$$

determines $X_T$ as a function of $S_T$. It gives us one more equation. This serves as a boundary condition.

# Euler Equation

- The difficult question is what would be the appropriate boundary condition in an infinite horizon problem. It is known that the following transversality condition is sufficient for an optimal solution:

$$\lim_{t\to\infty} \beta^t V'(S_t) S_t = -\lim_{t\to\infty} \beta^{t-1} \frac{r_1(X_{t-1}, S_{t-1})}{G_1(X_{t-1}, S_{t-1})} S_t = 0. \qquad (9)$$

  Since $V'(S_t)$ is the marginal value of $S_t$, $\beta^t V'(S_t) S_t$ is the present value of stock at $t$. The transversality condition implies that when $t$ goes infinite, the present value of stock must be negligible. That is, we should not be concerned about an infinite period later.

- Although Euler equation is not sufficient for the original problem, given usual technical conditions such as concavity etc, it is known that the Euler equation together with the transversality condition is sufficient for the original problem.

# Euler Equation

- *Example:* Consider the following problem.

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t U(c_t),$$
$$s.t.\ a_{t+1} = (1+\rho_t) a_t + w_t - c_t$$

- Bellman Equation

$$V(a_t, t) = \max_{c_t} \{U(c_t) + \beta V((1+\rho_t) a_t + w_t - c_t, t+1)\}$$

# Euler Equation

- FOC

$$U'(c_t) = \beta V_a(a_{t+1}, t+1)$$

- Envelope Theorem

$$V_a(a_t, t) = \beta(1+\rho) V_a(a_{t+1}, t+1)$$

# Euler Equation

- Euler equation:

$$\begin{aligned} \frac{U'\left(c\left(a_{t-1}, t-1\right)\right)}{\beta} &= U'\left(c\left(a_t, t\right)\right)\left(1+\rho_t\right) \\ \frac{U'\left(c_t\right)}{\beta U'\left(c_{t+1}\right)} &= 1+\rho_{t+1} \end{aligned}$$

- Transversality Condition

$$\lim_{t\to\infty} \beta^t V_a\left(a_t, t\right) a_t = \lim_{t\to\infty} \beta^t \frac{U'\left(c_{t-1}\right)}{\beta} a_t = \lim_{t\to\infty} \beta^{t-1} U'\left(c_{t-1}\right) a_t$$

- Euler equation is the same as the finite horizon problem, but the infinite horizon problem requires the transversality condition.

# Euler Equation

- **Assignment:** Suppose that $r(I_t, K_t) = \frac{1}{1+\rho_t}\left[r_t K_t - p_t\left(I_t + C(I_t, K_t)\right)\right]$, $K_{t+1} = G(I_t, K_t) = I_t + (1-\delta) K_t$ and $\beta_t = \frac{1}{1+\rho_t}$ where $K_t$ is capital stock, $I_t$ is investment, $r_t$ is the rental price of capital, $\delta$ is the depreciation rate, $p_t$ is the relative price of investment, $\rho_t$ is an real interest rate, $C_I(I_t, K_t) > 0$ $C_{II}(I_t, K_t) > 0$ and $C_K(I_t, K_t) < 0$. Suppose that there exists $B$ such that $\beta_t = \frac{1}{1+\rho_t} \leq B < 1$. (Note that discount factor is time dependent and that $\frac{1}{1+\rho_t} = \frac{\beta U'(c_{t+1})}{U'(c_t)}$ on the equilibrium from Euler equation by consumers.) A firm chooses $I$ to maximizes the present value of the stream of $r(I, K)$.
  1. Define the Bellman Equation.
  2. Derive the Euler Equation.
  3. Derive the Transversality Condition.

# The Neoclassical Growth Model

- We apply Guess and Verify method and Euler Equation to the Neoclassical Growth Model.
- We also explain the numerical method by using the Neoclassical Growth Model.

# The Neoclassical Growth Model

- Consider the previous neoclassical growth model:

$$\max_{\{C_t\}} \sum_{t=0}^{\infty} \beta^t N_t U\left(\frac{C_t}{N_t}\right),$$

$$\begin{aligned} s.t.\ K_{t+1} &= F\left(K_t, T_t N_t\right) + (1-\delta) K_t - C_t,\ \ K_0 \ \textit{is given} \\ T_{t+1} &= (1+g)\, T_t,\ \ T_0 \ \textit{is given} \\ N_{t+1} &= (1+n)\, N_t,\ \ N_0 \ \textit{is given} \end{aligned}$$

- First, we reformulate the model so that the analysis becomes simpler.

- Note that

$$
\begin{aligned}
K_{t+1} &= \left[ \begin{array}{c} F\left(\frac{K_t}{T_t N_t}, 1\right) + (1-\delta)\frac{K_t}{T_t N_t} \\ -\frac{C_t}{T_t N_t} \end{array} \right] T_t N_t, \\
\frac{K_{t+1}}{T_{t+1}N_{t+1}} \frac{T_{t+1}N_{t+1}}{T_t N_t} &= f\left(k_{et}\right) + (1-\delta) k_{et} - c_{et}, \\
k_{et+1}(1+g)(1+n) &= f\left(k_{et}\right) + (1-\delta) k_{et} - c_{et}, \\
k_{et+1} &= \frac{f\left(k_{et}\right) + (1-\delta) k_{et} - c_{et}}{(1+g)(1+n)}
\end{aligned}
$$

- Note also that

$$
\begin{aligned}
& \max_{\{C_t\}} \sum_{t=0}^{\infty} \beta^t N_t U\left(\frac{C_t}{N_t}\right) \\
= & \max_{\{C_t\}} \sum_{t=0}^{\infty} \beta^t (1+n)^t N_0 U\left(\frac{C_t}{T_t N_t} T_t\right) \\
= & N_0 \max_{\{c_{et}\}} \sum_{t=0}^{\infty} [\beta(1+n)]^t U(c_{et} T_t)
\end{aligned}
$$

# The Neoclassical Growth Model

- Hence, the original problem is equivalent to

$$\max_{\{c_{et}\}} \sum_{t=0}^{\infty} \hat{\beta}^t U(c_{et} T_t)$$

$$\begin{aligned} s.t.\ k_{et+1} &= \frac{f(k_{et}) + (1-\delta)k_{et} - c_{et}}{(1+g)(1+n)} \\ T_{t+1} &= (1+g)T_t,\ \textit{given}\ (k_0, T_0) \end{aligned}$$

- We assume that $\hat{\beta} = \beta(1+n) < 1$

# The Neoclassical Growth Model

- Bellman Equation

$$
\begin{aligned}
V\left(k_{et}, T_t\right) &= \max_{c_{et}} \left\{U\left(c_{et} T_t\right) + \hat{\beta} V\left(k_{et+1}, T_{t+1}\right)\right\} \\
k_{et+1} &= \frac{f\left(k_{et}\right) + \left(1-\delta\right) k_{et} - c_{et}}{\left(1+g\right)\left(1+n\right)} \\
T_{t+1} &= \left(1+g\right) T_t
\end{aligned}
$$

where $\hat{\beta} = \beta\left(1+n\right) < 1$.

- First Order Condition

$$U'\left(c_{et}T_t\right)T_t = \frac{\hat{\beta}V_k\left(k_{et+1}, T_{t+1}\right)}{\left(1+g\right)\left(1+n\right)}$$

- Envelope Theorem

$$V_k\left(k_{et}, T_t\right) = \frac{\hat{\beta}V_k\left(k_{et+1}, T_{t+1}\right)\left[f'\left(k_{et}\right)+\left(1-\delta\right)\right]}{\left(1+g\right)\left(1+n\right)}$$

# The Neoclassical Growth Model

- Euler Equation

$$\frac{U'\left(c_{et}T_t\right)T_t\left(1+g\right)\left(1+n\right)}{\hat{\beta}} = U'\left(c_{et+1}T_{t+1}\right)T_{t+1}\left[f'\left(k_{et+1}\right)+\left(1-\delta\right)\right]$$

Hence

$$\frac{U'\left(c_{et}T_t\right)\left(1+n\right)}{U'\left(c_{et+1}T_{t+1}\right)\beta\left(1+n\right)} = f'\left(k_{et+1}\right)+\left(1-\delta\right)$$

$$\frac{U'\left(c_{et}T_t\right)}{U'\left(c_{et+1}T_{t+1}\right)\beta} = f'\left(k_{et+1}\right)+\left(1-\delta\right)$$

- Balanced Growth: one of Kaldor's Stylized Facts (1963) suggests that the rate of return to capital, $f'(k_{et})$, is constant, and therefore, $k_{et} = k_{et+1} = k_e^*$ in the long run. Remember the dynamics of $k_{et}$ exhibits
$$k_e^* = \frac{f(k_e^*) + (1-\delta) k_e^* - c_{et}}{(1+g)(1+n)}.$$
This means $c_{et} = c_{et+1} = c_e^*$ in the long run. Euler equation implies that
$$\frac{U'(c_e^* T_t)}{U'(c_e^* T_{t+1}) \beta} = f'(k_e^*) + (1-\delta)$$
- Hence, if we wish to construct a model that can be consistent with Kaldor's Stylized Facts (1963), $\frac{U'(c_e^* T_t)}{U'(c_e^* T_{t+1})}$ must be constant

# The Neoclassical Growth Model

- In order to satisfy the empirical regularity that economy must exhibit a balanced growth, it is well-known that the following functional form is needed:

$$U\left(c_{et}T_t\right) = \frac{\left(c_{et}T_t\right)^{(1-\theta)} - 1}{1-\theta},\ 0 \leq \theta \neq 1$$

  This utility function is called the constant relative risk aversion utility function.

  - Remark: It can be shown that

$$\lim_{\theta\to 1} \frac{\left(c_{et}T_t\right)^{(1-\theta)} - 1}{1-\theta} = \ln\left(c_{et}T_t\right).$$

- Because $U'\left(c_e^* T_t\right) = \left(c_e^* T_t\right)^{-\theta}$

$$\frac{U'\left(c_e^* T_t\right)}{U'\left(c_e^* T_{t+1}\right)} = \frac{\left(c_e^* T_t\right)^{-\theta}}{\left(c_e^*\left(1+g\right)T_t\right)^{-\theta}} = \left(1+g\right)^{\theta}$$

- **Assignment:** Suppose that

$$U\left(c_{et}T_t\right)=\frac{\left(c_{et}T_t\right)^{(1-\theta)}-1}{1-\theta},\theta\geq 0$$

  - Show that $\theta$ can be interpreted as the measure of the constant relative risk aversion:

$$\theta=-\frac{U''\left(c_{et}T_t\right)c_{et}T_t}{U'\left(c_{et}T_t\right)}.$$

- It is also known that $\frac{1}{\theta}$ measures the elasticity of substitution between consumption at any two points in time.
- To see this, consider the following two period problems. Suppose that $U(C_t, C_{t+1}) = u(C_t) + \beta u(C_{t+1})$ and that the budget constraint is $p_t C_t + p_{t+1} C_{t+1} = W$, where $p_t$ and $p_{t+1}$ are prices at $t$ and $t+1$, and $W$ is the wealth. Since the first order condition is

$$\frac{p_{t+1}}{p_t} = \frac{U_2(C_t, C_{t+1})}{U_1(C_t, C_{t+1})} = \frac{\beta u'(C_{t+1})}{u'(C_t)},$$

- If $U(C_t) = \frac{(C_t)^{(1-\theta)}-1}{1-\theta}$, then $u'(C_t) = C_t^{-\theta}$. Hence,

$$\frac{p_{t+1}}{p_t} = \frac{\beta u'(C_{t+1})}{u'(C_t)} = \frac{\beta C_{t+1}^{-\theta}}{C_t^{-\theta}}$$

- It means that the elasticity of substitution, $-\frac{\frac{d\left(\frac{C_{t+1}}{C_t}\right)}{\frac{C_{t+1}}{C_t}}}{\frac{d\left(\frac{p_{t+1}}{p_t}\right)}{\frac{p_{t+1}}{p_t}}} = -\frac{d\ln\left(\frac{C_{t+1}}{C_t}\right)}{d\ln\left(\frac{p_{t+1}}{p_t}\right)}$, must satisfy the following condition.

$$-\frac{d\ln\left(\frac{C_{t+1}}{C_t}\right)}{d\ln\left(\frac{p_{t+1}}{p_t}\right)} = -\frac{d\ln\left(\frac{C_{t+1}}{C_t}\right)}{d\ln\left(\frac{\beta C_{t+1}^{-\theta}}{C_t^{-\theta}}\right)}$$

It is shown that

$$\begin{aligned}
-\frac{d\ln\left(\frac{C_{t+1}}{C_t}\right)}{d\ln\left(\frac{\beta C_{t+1}^{-\theta}}{C_t^{-\theta}}\right)} &= -\frac{d\left[\ln\left(C_{t+1}\right)-\ln\left(C_t\right)\right]}{d\left\{\ln\beta-\theta\left[\ln\left(C_{t+1}\right)-\ln\left(C_t\right)\right]\right\}} \\
&= -\frac{d\left[\ln\left(C_{t+1}\right)-\ln\left(C_t\right)\right]}{-\theta d\left[\ln\left(C_{t+1}\right)-\ln\left(C_t\right)\right]} = \frac{1}{\theta}.
\end{aligned}$$

- Suppose that $U(c_{et}T_t) = \frac{(c_{et}T_t)^{(1-\theta)}-1}{1-\theta}$. Then our current objective function, $\max_{\{c_{et}\}} \sum_{t=0}^{\infty} \hat{\beta}^t U(c_{et}T_t)$ can be further rewritten by a simpler expression.
- Note that when $\theta \neq 1$,

$$
\begin{aligned}
& \max_{\{c_{et}\}} \sum_{t=0}^{\infty} [\beta(1+n)]^t \frac{(c_{et}T_t)^{1-\theta}-1}{1-\theta} \\
= & \max_{\{c_{et}\}} \sum_{t=0}^{\infty} [\beta(1+n)]^t \frac{\left(c_{et}T_0(1+g)^t\right)^{1-\theta}-1}{1-\theta} \\
= & \max_{\{c_{et}\}} \left\{ \begin{array}{c} \sum_{t=0}^{\infty} [\beta(1+n)]^t (1+g)^{(1-\theta)t} \frac{(c_{et}T_0)^{1-\theta}}{1-\theta} \\ -\sum_{t=\tau}^{\infty} [\beta(1+n)]^t \frac{1}{1-\theta} \end{array} \right\} \\
= & \left\{ \begin{array}{c} T_0 \max_{\{c_{et}\}} \left[ \sum_{t=0}^{\infty} \left[\beta(1+n)(1+g)^{(1-\theta)}\right]^t \frac{(c_{et})^{1-\theta}}{1-\theta} \right] \\ -\sum_{t=\tau}^{\infty} \frac{[\beta(1+n)]^t}{1-\theta} \end{array} \right\}
\end{aligned}
$$

# The Neoclassical Growth Model

- Hence

$$
\begin{aligned}
& \arg\max_{\{c_{et}\}} \sum_{t=0}^{\infty} \left[\beta\left(1+n\right)\right]^{t} \frac{\left(c_{et}T_{t}\right)^{1-\theta}-1}{1-\theta} \\
= \quad & \arg\max_{\{c_{et}\}} \left\{ \sum_{t=0}^{\infty} \left[\beta\left(1+n\right)\left(1+g\right)^{(1-\theta)}\right]^{t} \frac{\left(c_{et}\right)^{1-\theta}}{1-\theta} \right\}
\end{aligned}
$$

- On the other hand, when $\theta = 1$, $U(c_{et} T_t) = \ln c_{et} T_t$. Hence

$$\begin{aligned}
& N_0 \max_{\{c_{et}\}} \sum_{t=0}^{\infty} [\beta (1+n)]^t \ln c_{et} T_t \\
= \quad & N_0 \max_{\{c_{et}\}} \sum_{t=0}^{\infty} [\beta (1+n)]^t \left[\ln c_{et} + \ln (1+g)^t + \ln T_0\right] \\
= \quad & N_0 \left\{ \begin{array}{c} \max_{\{c_{et}\}} \sum_{t=0}^{\infty} [\beta (1+n)]^t \ln c_{et} \\ + \sum_{t=0}^{\infty} [\beta (1+n)]^t [t \ln (1+g) + \ln T_0] \end{array} \right\}
\end{aligned}$$

- Hence

$$\arg\max_{\{c_{et}\}} N_0 \sum_{t=0}^{\infty} [\beta (1+n)]^t \ln c_{et} T_t = \arg\max_{\{c_{et}\}} \sum_{t=0}^{\infty} [\beta (1+n)]^t \ln c_{et}$$

- Hence, the original problem is equivalent to

$$\max_{\{c_{et}\}} \left\{ \sum_{t=0}^{\infty} (\beta^*)^t \, \tilde{u}(c_{et}) \right\}$$

$$s.t. \; k_{et+1} = \frac{f(k_{et}) + (1-\delta) k_{et} - c_{et}}{(1+g)(1+n)}, \; given \; k_0$$

where $\tilde{u}(c_{et}) = \frac{(c_{et})^{1-\theta}}{1-\theta}$ if $0 \le \theta \ne 1$ and $\ln(c_{et})$, if $\theta = 1$ and $\beta^* = \beta(1+n)(1+g)^{(1-\theta)}$.

- We assume that $\beta^* = \beta(1+n)(1+g)^{(1-\theta)} < 1$.

- We can define the Bellman equation as follows:

$$\begin{aligned} V(k_{et}) &= \max_{c_{et}} \left\{ \tilde{u}[c_{et}] + \beta^* V(k_{et+1}) \right\} \\ k_{et+1} &= \frac{f(k_{et}) + (1-\delta) k_{et} - c_{et}}{(1+g)(1+n)} \end{aligned}$$

where $\tilde{u}(c_{et}) = \frac{(c_{et})^{1-\theta}}{1-\theta}$ if $\theta \neq 1$ and $\ln(c_{et})$, if $\theta = 1$ and $\beta^* = \beta(1+n)(1+g)^{(1-\theta)}$.

- First Order Condition and Envelope Theorem

$$\begin{aligned}\tilde{u}'\left[c\left(k_{et}\right)\right] &= \frac{\beta^* V'\left(\kappa\left(k_{et}\right)\right)}{(1+g)(1+n)} \\ V'\left(k_{et}\right) &= \frac{\beta^* V'\left(\kappa\left(k_{et}\right)\right)\left[f'\left(k_{et}\right)+(1-\delta)\right]}{(1+g)(1+n)}\end{aligned}$$

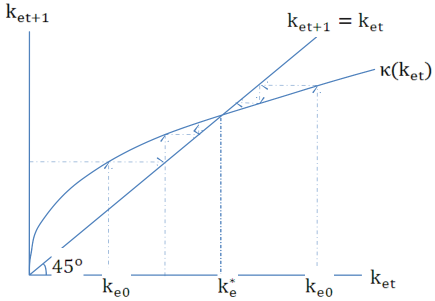
where

$$\kappa\left(k_{et}\right) = \frac{f\left(k_{et}\right)+(1-\delta)k_{et}-c\left(k_{et}\right)}{(1+g)(1+n)}$$

- **Assignment:** Suppose that $\theta = 1$ and therefore $\tilde{u}(c) = \ln c$, $\delta = 1$ and $f(k_e) = k_e^{\alpha}$. Show that

$$
\begin{aligned}
c(k_{et}) &= (1 - \alpha\beta^*)\, k_{et}^{\alpha} \\
\kappa(k_{et}) &= \frac{\alpha\beta k_{et}^{\alpha}}{(1+g)^{\theta}}
\end{aligned}
$$

## The Neoclassical Growth Model



# Euler Equation for Growth Model

- We cannot analyze guess and verify method for more general case. In this case, we can derive Euler equation.
- Remember that the first order condition and envelope theorem are

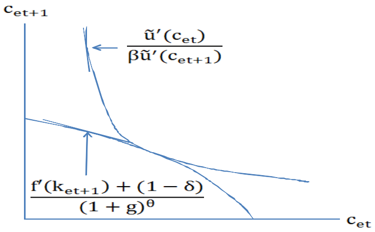
$$
\begin{aligned}
\tilde{u}'\left[c\left(k_{et}\right)\right] &= \frac{\beta^* V'\left(\kappa\left(k_{et}\right)\right)}{\left(1+g\right)\left(1+n\right)} \\
V'\left(k_{et}\right) &= \frac{\beta^* V'\left(\kappa\left(k_{et}\right)\right)\left[f'\left(k_{et}\right)+\left(1-\delta\right)\right]}{\left(1+g\right)\left(1+n\right)}
\end{aligned}
$$

# Euler Equation for Growth Model

- Hence, the Euler equation is

$$
\begin{aligned}
\frac{\tilde{u}'(c_{et-1})(1+g)(1+n)}{\beta^*} &= \tilde{u}'(c_{et})\left[f'(k_{et})+(1-\delta)\right]. \\
\frac{(1+g)(1+n)\tilde{u}'(c_{et})}{\beta(1+n)(1+g)^{(1-\theta)}\tilde{u}'(c_{et+1})} &= f'(k_{et+1})+(1-\delta) \\
\frac{\tilde{u}'(c_{et})}{\beta\tilde{u}'(c_{et+1})} &= \frac{f'(k_{et+1})+(1-\delta)}{(1+g)^{\theta}}
\end{aligned}
$$

# Euler Equation 2



# Euler Equation for Growth Model

- Note that $\tilde{u}(c_{et}) = \frac{(c_{et})^{1-\theta}}{1-\theta}$ means $\tilde{u}'(c_{et}) = (c_{et})^{-\theta}$. Hence the growth rate of the consumption is

$$\begin{aligned}
\frac{(c_{et})^{-\theta}}{(c_{et+1})^{-\theta}} &= \frac{\beta\left[f'(k_{et+1}) + (1-\delta)\right]}{(1+g)^{\theta}} \\
\left(\frac{c_{et+1}}{c_{et}}\right)^{\theta} &= \frac{\beta\left[f'(k_{et+1}) + (1-\delta)\right]}{(1+g)^{\theta}} \\
\frac{c_{et+1}}{c_{et}} &= \left[\frac{\beta\left[f'(k_{et+1}) + (1-\delta)\right]}{(1+g)^{\theta}}\right]^{\frac{1}{\theta}}
\end{aligned}$$

# Euler Equation for Growth Model

- One additional condition is the transversality condition:

$$\begin{aligned} 0 &= \lim_{t\to\infty} (\beta^*)^t V'(k_{et}) k_{et} \\ &= \lim_{t\to\infty} (\beta^*)^t \frac{\tilde{u}'[c_{et-1}]}{\beta^*} k_{et} (1+g)(1+n) \\ 0 &= \lim_{t\to\infty} [\beta^*]^{t-1} [c_{et-1}]^{-\theta} k_{et} \end{aligned}$$

Economic interpretation of this transversality condition is that it is not optimal to keep capital stock at the final date when the marginal value of consumption is positive. If so, the agent can always lower capital stock and consume more.

## Euler Equation for Growth Model

- Optimal Growth

$$\frac{c_{et+1}}{c_{et}} = \left[\frac{\beta\left[f'\left(k_{et+1}\right)+(1-\delta)\right]}{(1+g)^{\theta}}\right]^{\frac{1}{\theta}}$$

$$k_{et+1} = \frac{f\left(k_{et}\right)+(1-\delta)\,k_{et}-c_{et}}{(1+g)\,(1+n)}, \quad k_0 \text{ is given}$$

$$0 = \lim_{t\to\infty}\left[\beta^*\right]^{t-1}\left[c_{et-1}\right]^{-\theta} k_{et}$$

where $\beta^* = \beta\,(1+n)\,(1+g)^{(1-\theta)} < 1.$

- **Steady State Analysis:** the steady state is a point such that

$$c_e^* = c_{et+1} = c_{et}\,, k_e^* = k_{et+1} = k_{et}$$

- Note that if an economy reaches the steady state, the transversality condition is satisfied.

$$\begin{aligned} 0 &= \lim_{t\to\infty} \left[\beta^*\right]^{t-1} \left[c_e^*\right]^{-\theta} k_e^* \\ 0 &= \lim_{t\to\infty} \left[\beta^*\right]^{t-1} \end{aligned}$$

# Steady State Analysis

- Suppose that $\beta = \frac{1}{1+\varrho}$ where $\varrho$ is the discount rate. Then, on the steady state

$$
\begin{aligned}
\beta \left[ f'\left(k_e^*\right) + (1-\delta) \right] &= (1+g)^\theta \\
f'\left(k_e^*\right) &= \frac{(1+g)^\theta}{\beta} - (1-\delta) \\
&= (1+g)^\theta (1+\varrho) - (1-\delta)
\end{aligned}
$$

and

$$
\begin{aligned}
k_e^* &= \frac{f\left(k_e^*\right) + (1-\delta) k_e^* - c_e^*}{(1+g)(1+n)} \\
c_e^* &= f\left(k_e^*\right) + (1-\delta) k_e^* - (1+g)(1+n) k_e^* \\
&= f\left(k_e^*\right) - \left[ (1+g)(1+n) - (1-\delta) \right] k_e^*
\end{aligned}
$$

# Steady State Analysis

- **Golden Rule:** Note

$$\frac{dc_e^*}{dk_e^*} = f'(k_e^*) - [(1+g)(1+n) - (1-\delta)]$$
$$\frac{d^2 c_e^*}{d(k_e^*)^2} = f''(k_e^*) < 0$$

Hence

$$f'\left(k_e^{GR}\right) = (1+g)(1+n) - (1-\delta)$$

- Compare with

$$f'(k_e^*) = \frac{(1+g)^\theta}{\beta} - (1-\delta)$$

# Steady State Analysis

- Because $\beta^* = \beta (1+g)^{1-\theta} (1+n) < 1$,

$$
\begin{aligned}
& (1+g)(1+n) - (1-\delta) - \left[ \frac{(1+g)^{\theta}}{\beta} - (1-\delta) \right] \\
= \quad & (1+g)(1+n) - \frac{(1+g)^{\theta}}{\beta} \\
= \quad & \left[ \beta (1+g)^{1-\theta} (1+n) - 1 \right] \frac{(1+g)^{\theta}}{\beta} < 0.
\end{aligned}
$$

- Hence

$$
f'\left(k_e^{GR}\right) < f'\left(k_e^*\right) \Rightarrow k_e^* < k_e^{GR}
$$

  - Because people discount future, the steady state value of capital stock is lower than the golden rule level of capital stock. The steady state level of capital stock is called the modified golden rule level of capital stock.

# Steady State Analysis

- Saving rate:

$$\begin{aligned} s &= \frac{Y_t - C_t}{Y_t} \\ &= \frac{\left[f\left(k_e^*\right) - c_e^*\right] T_t N_t}{f\left(k_e^*\right) T_t N_t} \\ &= \frac{f\left(k_e^*\right) - \left[f\left(k_e^*\right) - \left(\delta + g + n + ng\right) k_e^*\right]}{f\left(k_e^*\right)} \\ &= \frac{\left(\delta + g + n + ng\right) k_e^*}{f\left(k_e^*\right)} \end{aligned}$$

Hence the constant saving rate is supported around the steady sate.

# Steady State Analysis

- Assume $f(k) = k^{\alpha}$.

$$
\begin{aligned}
s &= \frac{(\delta + g + n + ng)\, k_e^*}{\left(k_e^*\right)^{\alpha}} \\
\left(k_e^*\right)^{1-\alpha} &= \frac{s}{\delta + g + n + ng} \\
k_e^* &= \left[\frac{s}{\delta + g + n + ng}\right]^{\frac{1}{1-\alpha}}.
\end{aligned}
$$

This condition is the same as that in Solow model.

# Steady State Analysis

- Now we can endogenize $s$. Note that $f'(k) = \alpha k^{\alpha-1}$ and that on the steady state,

$$\begin{aligned} f'(k_e^*) &= (1+g)^{\theta}(1+\varrho) - (1-\delta) \\ \alpha (k_e^*)^{\alpha-1} &= (1+g)^{\theta}(1+\varrho) - (1-\delta), \\ (k_e^*)^{1-\alpha} &= \frac{\alpha}{(1+g)^{\theta}(1+\varrho) - (1-\delta)}, \end{aligned}$$

Hence

$$\begin{aligned} s &= (\delta + g + n + ng)(k_e^*)^{1-\alpha} \\ &= \frac{\alpha(\delta + g + n + ng)}{(1+g)^{\theta}(1+\varrho) - (1-\delta)} \\ &= \frac{\alpha[(1+g)(1+n) - (1-\delta)]}{(1+g)^{\theta}(1+\varrho) - (1-\delta)} \end{aligned}$$

# Steady State Analysis

- Note that saving rate is larger when we have more population growth. When you expect to have more children in future, you will have move incentive to save.
- When $\varrho$ is large, the agent largely discounts his future. Since today is more important than tomorrow, the agent saves less.
- Similarly, if $g > 0$, the larger the marginal rate of substitution $\frac{1}{\theta}$, the saving rate is. When $\theta$ is small, the marginal rate of substitution is large. The agent is willing to change his consumption in response to the changes in the return. When the economy is growing, the return is high. Therefore, this behavior implies that the agent saves more.

- The steady state value of capital stop per unit of effective labor is.

$$k_e^* = \left[\frac{1}{\delta+g+n+ng}\frac{\alpha\left(\delta+g+n+ng\right)}{\left(1+g\right)^{\theta}\left(1+\varrho\right)-\left(1-\delta\right)}\right]^{\frac{1}{1-\alpha}},$$

$$= \left[\frac{\alpha}{\left(1+g\right)^{\theta}\left(1+\varrho\right)-\left(1-\delta\right)}\right]^{\frac{1}{1-\alpha}},$$

- Using that $y_e^* = \left(k_e^*\right)^{\alpha}$

$$\frac{Y_t}{N_t} = \left[\frac{\alpha}{\left(1+g\right)^{\theta}\left(1+\varrho\right)-\left(1-\delta\right)}\right]^{\frac{\alpha}{1-\alpha}} T_t.$$

# Steady State Analysis

- Different from Solow model, population growth has no impact on the GDP per capita. Because the saving rate is an increasing function of population growth, the negative impacts of population growth on GDP per capita is canceled out.
- The more the agent discount the future, (large $\varrho$), the lower the saving rate and, therefore, the lower the steady state value of per capita GDP.
- Hence, if $g > 0$, the larger the marginal rate of substitution $\frac{1}{\theta}$, the larger per capita GDP is.

# Steady State Analysis

- **Assignment:** Derive the consumption per capita on the steady state.

# Numerical Analysis

- In order to obtain a numerical solution, many economists use Matlab.
- Although this may not be the fastest program, it is instructive to demonstrate how we can write Matlab code to obtain a numerical policy function and numerical value function of the neoclassical growth model.
- I first explain the basic code of Matlab and later I use Matlab to derive a numerical policy function and numerical value function.

# Numerical Analysis

- **Turning on Matlab:** To access Matlab, you can go to the network center.
  - If you cannot go to the network center, you can alternatively use GNU Octave (basically a free clone of MATLAB).
  - In order to obtain GNU Octave, go to
    - https://www.gnu.org/software/octave/.
  - If you want to know how to use GNU Octave, go to
    - http://www.struct.t.u-tokyo.ac.jp/katsu/lectures/optimization/octave-digest.pdf
    - https://en.wikibooks.org/wiki/Octave_Programming_Tutorial
    - https://octave.org/doc/v4.0.1/

# Numerical Analysis

- **Getting Help in Matlab:** If you have questions about the use of a particular command, you just type
  - *help [command]*
    - **Example:** the command "plot" plots variables. If you want help on how to use it, you would type
      *help plot*

# Numerical Analysis

- **Basic Math in Matlab:**
  - Symbols for basic operations are "+" for addition, "-" for subtraction, "*" for multiplication and "/" for division. "^" is "to the power of." If you want Matlab to perform a computation, just type it.
    - **Example:** if you want Matlab to compute "2+2," just type into the command line
      *2+2*
  - If you want to assign a value to a variable, just type it out.
    - **Example:** if you want the variable "apple" to have the value 3 and the variable "orange" to have the value 5, then type
      *apple=3*
      *orange=5*

- **Basic Math in Matlab:**
  - Matlab will remember these values unless you change them. Computations with variables are just as simple as with numbers.
    - **Example:** if you want to find out what apple+orange is, you just type *apple+orange*
  - You will notice that when you type something like "apple=3," the computer repeats it. To stop that from happening, just put a semi-colon at the end.
    - **Example:** If you don't want the computer to repeat what you have done, type *apple=3;*

# Numerical Analysis

- **Vectors and Matrices:** For the most part, in Matlab you will be programming with vectors and matrices, not just numbers. This is no big stretch, however. After all, a variable is just a $1 \times 1$ matrix. Matlab uses the notation $X(i,j)$ to refer to the element in the $i$th row and the $j$th column of a matrix $X$.
  - **Example:** if you have a matrix called "apple" and you want the 3rd element of the second row to equal 4.5, you would type *apple(2,3)=4.5;*
  - If you use Matlab in the network center, it is a bit old. In this case, if you haven't told Matlab that you want a matrix called apple that is at least 2x3, however, it may give you an error. Therefore, it is a good idea to initialize all matrices at the beginning of the program before you actually use them. This is called "declaring" a variable. One way to do this is to write *apple=zeros(2,3);*
  - Matlab creates the 2x3 matrix apple, and sets all its elements to equal zero.

- **Loops in Matlab:** All programming languages of which I am aware make extensive use of something called "loops". This is a way of repeating a certain task many times or under certain conditions.
- **for... end:** This is the classic loop.
  - **Example:** Suppose you wanted to create a 101x1 vector called "orange", where the elements are the numbers from zero to 1 in increments of 0.01. You would do the following:
    - ```
      for i=1:101
        orange(i)=(i-1)/100;
      end
      ```
  - Matlab begins by setting i=1, then proceeds until it hits the word "end". Then it returns to the "for" line, sets i=2, and repeats this process until i=101.
      ```

- You can have multiple concentric loops.
  - **Example:** suppose that for some reason you want to have a 100x100 matrix $X$ where $X_{ij} = i^2 + j^3$. Then you would use:
    - 
      ```
      X=zeros(100,100);
        for i=1:100
          for j=1:100
              X(i,j)=i^2+j^3;
          end
        end
      ```
  - Notice that what is inside each loop is indented. When you have a lot of loops it can be hard to keep track of things, and indentation is a way of telling where you are.

# Numerical Analysis

- **if... end:** Another loop-like structure involves conditional statements.
  - **Example:** Suppose you have generated a vector of 100 numbers called $X$. You want to see whether they are positive or negative, and store that information in another vector called $Y$. You could proceed as follows.
    - ```
      Y=zeros(100,1);
       for i=1:100
          if X(i)>0
             Y(i)=1;
          end
       end
      ```
  - When it reads "if", Matlab checks whether the condition is met (X(i)>0) and if so performs what comes afterwards until it hits "end." If the condition is not met, it just ignores everything until "end".
      ```

# Numerical Analysis

- An alternative is
  - ```
    Y=zeros(100,1);
    for i=1:100
        if X(i)>0
            Y(i)=1;
        else
            Y(i)=0;
        end
    end
    ```
- Here Matlab does the same as before, except that if the condition is not met it does not ignore everything, rather it executes the commands that lie after "else" instead. In this example, of course, that is redundant (since the default value of $Y$ is zero).
    ```

- **Writing your own Programs and Functions in Matlab:**
  - **Programs:** A "program" is nothing but a list of commands (such as those above) that you save so that you can use them over and over again. How do you write programs? Just find any text editor, and type out the sequence of commands that you desire. Then, save it to a disk
    - **Example:** You can save your program to a file such as "takii.m" If you want to run takii.m, just type
      *takii*
      Then Matlab will perform all the commands in the file, in order.
  - There is a function in Matlab called "save." However, "save" does not save programmes, rather it saves the values of all the variables in the memory.
    - **Example:** if you type:
      *save katsuya*
      Matlab will create a file called katsuya.mat. If for some reason you wanted to retrieve the values of the variables in that file, you would type:
      *load katsuya*

- **Functions:** Sometimes you may find it convenient to write down a function that does not exist in Matlab. Then you can use a command "function".
  - **Example:** This would be a file takii.m:
    - *function [z] = takii(x)*
      *z = (x+1)^2;*
      After saving this file, if you input "takii(2)" into Matlab you should get "9" as output: takii.m is now a function. In that program, $x$ represents the input and $z$ represents the output.

- **Solving equations:** Matlab can also help us to solve non-linear equations.
  - **Example:** The following line embedded in another program finds the zeroes of 'takii', or $\{x : takii(x) = 0\}$:
    - *fun=@takii*
      *x=10*
      *fsolve(fun,x)*

    Matlab starts at $x = 10$ and uses gradient methods to find the value of x such that takii($x$) is zero. In our example, $x = -1$.
  - Why would you want to do this? Many economic problems are solved by finding values of some variable $x$ such that $f(x) = 0$ for some function $f$. For example, the deterministic steady state in the neoclassical growth model is defined in this manner. Fsolve is one way to find it..

- **Comments:** Good programming style follows certain conventions so that others can easily pick up your code and see what you are doing. For this purpose, it is important to write comments on your program.
  - **Example:** If you want to make a comment, "This is where I compute the steady state." you would type:
    *% Here I compute the steady state*
    Matlab ignores everything in a line after it sees "%", so you can write anything there that you wish.

- We are now ready to write a program to derive a policy function and a value function of the growth model expressed by the following Bellman equation.

$$
\begin{aligned}
V(k_{et}) &= \max_{c_{et}} \left\{ \tilde{u}\left[c_{et}\right] + \beta^* V(k_{et+1}) \right\} \\
k_{et+1} &= \frac{k_{et}^{\alpha} + (1-\delta) k_{et} - c_{et}}{(1+g)(1+n)}
\end{aligned}
$$

where $\tilde{u}(c_{et}) = \frac{(c_{et})^{1-\theta}}{1-\theta}$ if $\theta \neq 1$, $\ln(c_{et})$ if $\theta = 1$, and $\beta^* = \frac{(1+n)(1+g)^{(1-\theta)}}{1+\varrho}$.

- I explain the process with the help of the LECTURE NOTE OF RECURSIVE MACROECONOMICS by MARTIN ELLISON. The following argument is based on his lecture note.
- There are four main parts to the code:
  1. parameter value initialization,
  2. state space definitions,
  3. value function iterations
  4. output.

- **Parameter value initialization:** Start new program by clearing the work space. Parameters are initialized with *theta* the coefficient of relative risk aversion, *discount* the discount rate, *alpha* the exponent on capital in the production function, *g* the growth rate of productivity, *n* the population growth rate, *beta* discount factor and *delta* the depreciation rate.
  - ```
    clear all
    theta=1.2;
    discount=.05;
    alpha=.3;
    g=0.02;
    n=0.02;
    beta=((1+n)*((1+g)^(1-theta)))/(1+discount);
    delta=.05;
    ```
    ```

- **State space definitions:** We now construct a grid of possible capital stocks. The grid is centered on the steady-state capital stock per unit of effective labor, with maximum ( ) and minimum ( ) values 105% and 95% of the steady-state value. There are $N$ discrete points in the grid. Note that

$$k_e^* = \left[ \frac{\alpha}{(1+g)^{\theta}(1+\varrho) - (1-\delta)} \right]^{\frac{1}{1-\alpha}}.$$

  - *N=1000;*
    *Kstar=(alpha/(((1+g)^(theta))*(1+discount)-(1-delta)))^(1/(1-alpha));*
    *Klo=Kstar*0.95;*
    *Khi=Kstar*1.05;*
    *step=(Khi-Klo)/N;*
    *K=Klo:step:Khi;*
    *l=length(K);*

- $K$ is the vector.

$$K = \underbrace{[Klo, Klo + step, Klo + 2step, ..., Khi]}_{I}$$

# Numerical Analysis

- **Value function iterations:** Value function iteration requires 3 steps
  1. Set initial guess for the value function.
  2. Express one period return function as a function of current state variable and future state variable.
     - In this way, the Bellman equation can be expressed only by current state variable and future state variable.
  3. Iterations
     - Add one period return function to the discounted initial guess and find the optimal value. This value serves the initial guess in the next round. Iterate it until the initial guess meets the optimal value.

- **Initial Guess for the value function:** initial guess is taken by assuming that all capital is always consumed immediately. In this case,

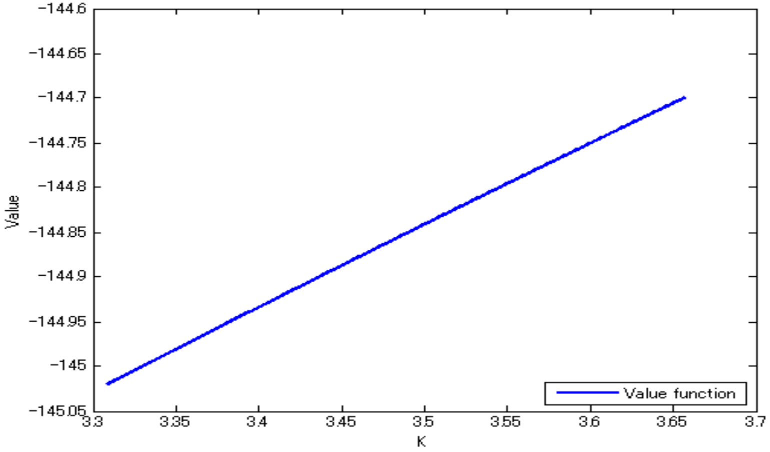
$$\begin{aligned} v_0 &= \ln(ytot) \;\; if \;\; \theta = 1 \\ v_0 &= \frac{(ytot)^{1-\theta}}{1-\theta}, \;\; if \;\; \theta \neq 1. \\ ytot &= k_{et}^{\alpha} + (1-\delta)\, k_{et}. \end{aligned}$$

- Hence, our initial guess is written as
  - ```
    ytot = K.^alpha+(1-delta)*K;
    ytot=ytot'*ones(1,l);
    if theta==1
    v=log(ytot);
    else
    v=ytot.^(1-theta)/(1-theta);
    end
    ```
    where *ones(1,l)*.are $1 \times l$ vector of 1 and the syntax *.^* defines element-by-element operation, so each element of a vector or matrix is risen to the power of the exponent.
    ```

# Numerical Analysis

- Note that $v$ is $I \times I$ matrix with every column being the same. For example, if $\theta \neq 1$,

$$v = \begin{bmatrix} \frac{\left[(Klo)^{\alpha}+(1-\delta)Klo\right]^{1-\theta}}{1-\theta} & \frac{\left[(Klo)^{\alpha}+(1-\delta)Klo\right]^{1-\theta}}{1-\theta} & \frac{\left[(Klo)^{\alpha}+(1-\delta)Klo\right]^{1-\theta}}{1-\theta} \\ .. & .. & .. \\ \frac{\left[(Khi)^{\alpha}+(1-\delta)Khi\right]^{1-\theta}}{1-\theta} & \frac{\left[(Khi)^{\alpha}+(1-\delta)Khi\right]^{1-\theta}}{1-\theta} & \frac{\left[(Khi)^{\alpha}+(1-\delta)Khi\right]^{1-\theta}}{1-\theta} \end{bmatrix}$$

- **Express one period return function as a function of current state variable and future state variable.** For use in the value function iterations, we define an $I \times I$ matrix $C$

$$\begin{aligned}
C &= \begin{bmatrix} c_e(1,1) & \dots & c_e(1,I) \\ .. & .. & .. \\ c_e(I,1) & .. & c_e(I,I) \end{bmatrix} \\
c_e(i,j) &= k_{et}^{\alpha}(j) + (1-\delta)\,k_{et}(j) - (1+g)(1+n)\,k_{et+1}(i) \\
k_{et}(j) &= Klo + (j-1) \times step, \\
k_{et+1}(i) &= Klo + (i-1) \times step
\end{aligned}$$

- The program for this operation is
  - *for i=1:I*
    *for j=1:I*
    *C(i,j) = K(j).^alpha + (1-delta)*K(j) - (1+g)*(1+n)*K(i);*
    *end*
    *end*

- Next, we need to map consumption levels into utility levels by applying the suitable utility function. The syntax ./ defines element-by-element operation rather than matrix division implied by using the / operator.
  - ```
    if theta==1
      U=log(C);
    else
      U=(C.^(1-theta))/(1-theta);
    end
    ```
    ```

- **Iteration:** Now we can use our first guess of the value function to get the second estimate *v1*, which will be the start of our next $T$ iterations.
    - ```
      T=100
      for j=1:T
         w=U+beta*v;
          v1=max(w);
          v=v1' *ones(1,l);
      end
      ```
      ```

# Numerical Analysis

- Note that $w$ is the $I \times I$ matrix. Suppose $\theta \neq 1$. Then

$$
\begin{aligned}
w &= \begin{bmatrix} \frac{[c_e(1,1)]^{1-\theta}}{1-\theta} + \beta^* v(1,1) & \dots & \frac{[c_e(1,I)]^{1-\theta}}{1-\theta} + \beta^* v(I,I) \\ .. & .. & .. \\ \frac{[c_e(I,1)]^{1-\theta}}{1-\theta} + \beta^* v(I,1) & .. & \frac{[c_e(I,I)]^{1-\theta}}{1-\theta} + \beta^* v(I,I) \end{bmatrix} \\
c_e(i,j) &= k_{et}^{\alpha}(j) + (1-\delta) k_{et}(j) - (1+g)(1+n) k_{et+1}(i) \\
v(i,j) &= \frac{\left[[k_{et+1}(i)]^{\alpha} + (1-\delta) k_{et+1}(i)\right]^{1-\theta}}{1-\theta} \\
k_{et}(j) &= Klo + (j-1) \times step, \\
k_{et+1}(i) &= Klo + (i-1) \times step
\end{aligned}
$$

- Hence, $v1$ find the maximum value for each column.

$$v1 = \left[ \begin{array}{ccc} \frac{\left[c_e\left(i_1^*,1\right)\right]^{1-\theta}}{1-\theta} + \beta^* v\left(i_1^*, 1\right) & .. & \frac{\left[c_e\left(i_I^*,I\right)\right]^{1-\theta}}{1-\theta} + \beta^* v\left(i_I^*, I\right) \end{array} \right]$$

where $i_j^*$ is the maximum indices of the $j$ th column.

- Hence, $v$ is again, $I \times I$ matrix with every column being the same. For example, if $\theta \neq 1$,

$$v = \left[ \begin{array}{ccc} \frac{\left[c_e\left(i_1^*,1\right)\right]^{1-\theta}}{1-\theta} + \beta^* v\left(i_1^*, 1\right) & .. & \frac{\left[c_e\left(i_1^*,1\right)\right]^{1-\theta}}{1-\theta} + \beta^* v\left(i_1^*, 1\right) \\ .. & .. & .. \\ \frac{\left[c_e\left(i_I^*,I\right)\right]^{1-\theta}}{1-\theta} + \beta^* v\left(i_I^*, I\right) & .. & \frac{\left[c_e\left(i_I^*,I\right)\right]^{1-\theta}}{1-\theta} + \beta^* v\left(i_I^*, I\right) \end{array} \right]$$

- **Output:** Value function iterations are now complete. The final value function is stored in *Val* and the indices of the future $k_{et+1}$ are stored in *ind*. These latter indices are converted into $k_{et+1}$ values in *optk*. Using *optk*, we can create the optimal value of consumption, *optc*.
  - ```
    [val,ind]=max(w);
    optk = K(ind);
    optc = K.^alpha + (1-delta)*K - (1+g)*(1+n)*optk;
    ```
    ```

- To illustrate the results, I first plot the value function.
  - *figure(1)*
    *plot(K,v1','LineWidth',2)*
    *xlabel('K');*
    *ylabel('Value');*
    *legend('Value function',4);*

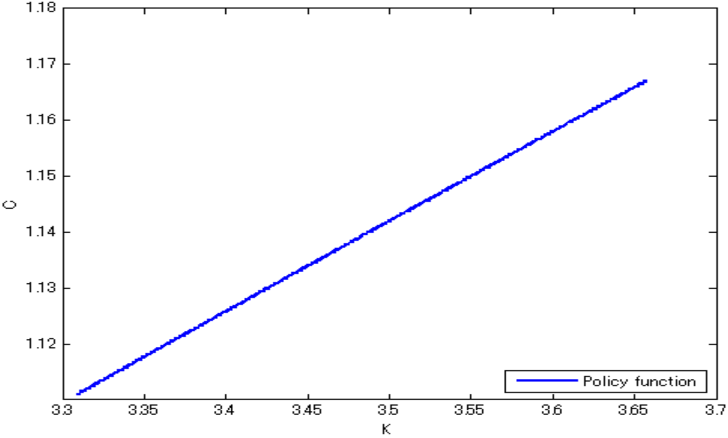
# Numerical Analysis



# Numerical Analysis

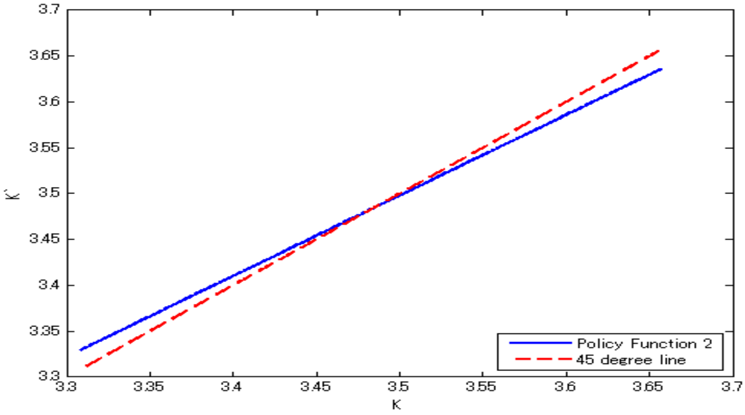
- Next I plot the policy function
  - *figure(2)*
    *plot(K,optc','LineWidth',2)*
    *xlabel('K');*
    *ylabel('C');*
    *legend('Policy function',4);*

# Numerical Analysis



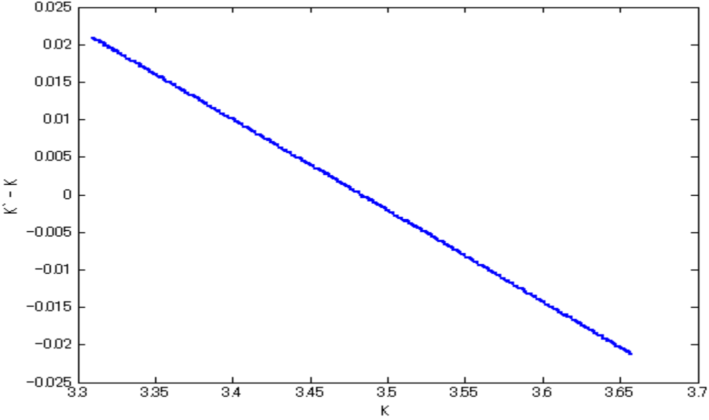
- It is also useful to plot how $k_{et+1}$ varies as a function of $k_{et}$. It shows that an economy converges to the steady state.
  - *figure(3)*
    *plot(K,optk','LineWidth',2)*
    *hold on*
    *plot(K,K','–r','LineWidth',2)*
    *xlabel('K');*
    *ylabel('K'');*
    *legend('Policy Function 2','45 degree line',4);*

# Numerical Analysis



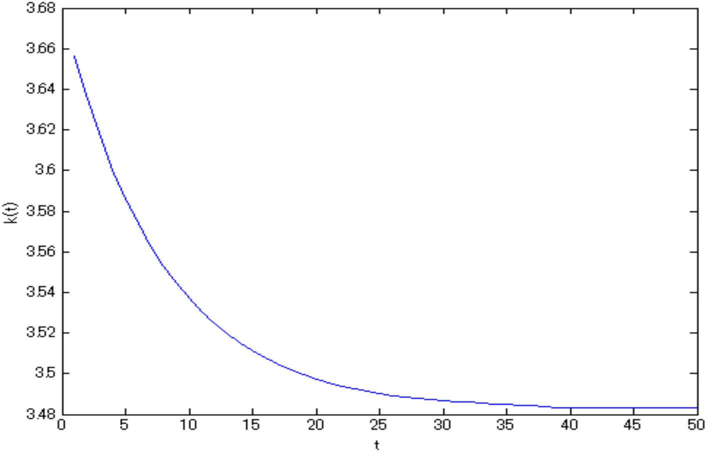
- Another useful figure is showing net investment $k_{et+1} - k_{et}$ as a function of $k_{et}$. The figure confirms that net investment is positive when $k_{et} < k_e^*$ and negative when $k_{et} > k_e^*$.
  - *figure(4)*
    *plot(K,(optk-K)','LineWidth',2)*
    *xlabel('K');*
    *ylabel('K' - K');*

# Numerical Analysis



- To gain an insight into the dynamics of the adjustment process, it is useful to calculate the path of convergence of capital if initial capital is not equal to its steady-state value. $p$ is the number of periods over which to follow convergence. The indices of the capital choices are stored in the $p \times 1$ vector $mi$ and the capital stocks themselves are stored in the $p \times 1$ vector $m$. The initial index is taken as 1 so the initial capital stocks are at its maximum value.
  - ```
    p=50;
    mi=zeros(p,1);
    m=zeros(p,1);
    mi(1)=N;
    m(1)=K(mi(1));
    for i=2:p
     mi(i)=ind(mi(i-1));
     m(i) = K(mi(i));
    end
    ```
    ```

- We can plot this dynamics by the following program.
  - ```
    t=1:1:50
    figure(5)
    plot(t,m)
    xlabel('t');
    ylabel('k(t)');
    ```
    ```

# Numerical Analysis



- **Assignment:** Replicate what I have done in this lecture.

# Recursive Competitive Equilibrium

- **Market Economy:** This section introduces a recursive competitive equilibrium. Recursive competitive equilibrium is a particularly convenient because

  1. it naturally fits the dynamic programing approach.
  2. it is easily applied in a wide variety of settings, including those with distortions.
  3. the equilibrium process can be computed and can be simulated to general equilibrium paths for the economy.

- We ask how does this command economy relate to the market economy? In order to answer this question, remember that the first and second welfare theorem. The first welfare theorem says that the market economy is Pareto optimal under some conditions. The second welfare theorem says that the Pareto optimum allocation can be supported by a market economy with a income transfer under some conditions. Note that our command economy is Pareto optimum and since every agent is identical, there is no reason to consder a income transfer in our model. Hence, there must have prices which support our equilibrium.

- In this section, we show that our economy can be supported by a market economy. In particular, we show that there is a price mechanism that can reproduce a following policy function and a value function that satisfy the first order condition and envelope theorem of our problem.

$$\tilde{u}'\left[c\left(k_{et}\right)\right] = \frac{\beta^* V'\left(\kappa\left(k_{et}\right)\right)}{\left(1+g\right)\left(1+n\right)},$$

$$V'\left(k_{et}\right) = \frac{\beta^* V'\left(\kappa\left(k_{et}\right)\right)\left[f'\left(k_{et}\right)+\left(1-\delta\right)\right]}{\left(1+g\right)\left(1+n\right)},$$

where

$$\kappa\left(k_{et}\right) = \frac{f\left(k_{et}\right)+\left(1-\delta\right)k_{et}-c\left(k_{et}\right)}{\left(1+g\right)\left(1+n\right)}.$$

- Let us first consider the following representative household problem

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t N_t U(c_t)$$

$$\begin{aligned} s.t.\ A_{t+1} &= (1+\rho(k_{et}))A_t + w(k_{et})T_t N_t - c_t N_t,\ A_0 = k_{e0}T_0 N_0 \\ k_{et+1} &= \frac{f(k_{et}) + (1-\delta)k_{et} - C^m(k_{et})}{(1+g)(1+n)},\ k_{e0} \text{ is given} \\ T_{t+1} &= (1+g)T_t,\ T_0 \text{ is given} \\ N_{t+1} &= (1+n)N_t,\ N_0 \text{ is given} \end{aligned}$$

  where $c_t$ is consumption per capita, $A_t$ is the asset of this household, $k_{et}$, $T_t$ and $N_t$ are aggregate state variables.
- Note that the interest rate $\rho$, and the wage per unit of effective labor, $w$ are function of the aggregate capital per unit of effective labor, $k_{et}$.

- We also assume that an initial individual asset is the same as the aggregate capital, $A_0 = k_{e0}T_0 N_0$.

# Recursive Competitive Equilibrium

- **Assignment:** Suppose that

$$U\left(c_{et}T_t\right)=\frac{\left(c_{et}T_t\right)^{(1-\theta)}-1}{1-\theta},\theta\geq 0$$

Show that the original problem is equivalent to

$$\max_{\{c_{et}\}}\sum_{t=0}^{\infty}\left(\beta^*\right)^t\tilde{u}\left(c_{et}\right)$$

$$a_{et+1} = \frac{\left(1+\rho\left(k_{et}\right)\right)a_{et}+w\left(k_{et}\right)-c_{et}}{\left(1+g\right)\left(1+n\right)},\quad a_0=k_{e0}$$

$$s.t.\ k_{et+1} = \frac{f\left(k_{et}\right)+\left(1-\delta\right)k_{et}-C^m\left(k_{et}\right)}{\left(1+g\right)\left(1+n\right)},\quad k_{e0}\ is\ given$$

where $\tilde{u}\left(c_{et}\right)=\frac{\left(c_{et}\right)^{1-\theta}}{1-\theta}$ if $0\leq\theta\neq 1$, $\ln\left(c_{et}\right)$ if $\theta=1$, and $\beta^*=\beta\left(1+n\right)\left(1+g\right)^{(1-\theta)}$.

- Bellman Equation

$$
\begin{aligned}
V^h\left(a_{et}, k_{et}\right) &= \max_{c_{et}} \left\{ \tilde{u}\left(c_{et}\right) + \beta^* V^h\left(a_{et+1}, k_{et+1}\right) \right\} \\
a_{et+1} &= \frac{\left(1+\rho\left(k_{et}\right)\right) a_{et} + w\left(k_{et}\right) - c_{et}}{\left(1+g\right)\left(1+n\right)} \\
k_{et+1} &= \frac{f\left(k_{et}\right) + \left(1-\delta\right) k_{et} - C^m\left(k_{et}\right)}{\left(1+g\right)\left(1+n\right)}
\end{aligned}
$$

- First order conditions

$$\tilde{u}'(c(a_{et}, k_{et})) = \frac{\beta^* V_1^h(a_{et+1}, k_{et+1})}{(1+g)(1+n)}$$

where $c(a_{et}, k_{et})$ is a policy function

- Envelope Theorem

$$V_1^h(a_{et}, k_{et}) = \frac{\beta^* V_1^h(a_{et+1}, k_{et+1})[1+\rho(k_{et})]}{(1+g)(1+n)}$$

# Recursive Competitive Equilibrium

- Assume that a representative firm maximizes capital and labor given a factor price functions

$$
\begin{aligned}
& \max_{K_t^f, L_t} \left\{ F\left(K_t^f, T_t L_t\right) - r\left(k_{et}\right) K_t^f - w\left(k_{et}\right) T_t L_t \right\}, \\
= \quad & T_t \max_{k_{et}^f, L_t} \left\{ F\left(k_{et}^f, 1\right) - r\left(k_{et}\right) k_{et}^f - w\left(k_{et}\right) \right\} L_t \\
= \quad & T_t \max_{k_{et}^f, L_t} \left\{ f\left(k_{et}^f\right) - r\left(k_{et}\right) k_{et}^f - w\left(k_{et}\right) \right\} L_t
\end{aligned}
$$

where $k_{et}^f = \frac{K_t^f}{T_t L_t}$. The rental price $r$ is also the function of the aggregate capital per unit of effective labor.
- The first order conditions are

$$
\begin{aligned}
r\left(k_{et}\right) &= f'\left(k_e^f\left(k_{et}\right)\right), \\
w\left(k_{et}\right) &= f\left(k_e^f\left(k_{et}\right)\right) - r\left(k_{et}\right) k_e^f\left(k_{et}\right)
\end{aligned}
$$

# Recursive Competitive Equilibrium

- Arbitrage condition

$$\rho\left(k_{et}\right)=r\left(k_{et}\right)-\delta$$

- Labor and Capital Market

$$\begin{aligned} L_t &= N_t, \\ k_e^f\left(k_{et}\right)T_tL_t &= a_{et}T_tN_t \end{aligned}$$

  - We assume that the initial value of individual asset is the same as that of the aggregate asset.

$$a_{et}T_tN_t=k_{et}T_tN_t$$

  - Two market conditions with this initial condition imply

$$k_e^f\left(k_{et}\right)=k_{et}=a_{et}.$$

# Recursive Competitive Equilibrium

**Definition:** Recursive competitive equilibrium consists of a value function $V_1^h(a_{et}, k_{et})$, a policy function $c(a_{et}, k_{et})$, a corresponding set of aggregate decision $C^m(k_{et})$, production plans $k_e^f(k_{et})$, set of factor price functions, $\{r(k_{et}), w(k_{et}), \rho(k_{et})\}$ such that these functions satisfy

- Household

$$\begin{aligned} \tilde{u}'(c(a_{et}, k_{et})) &= \frac{\beta^* V_1^h(a_{et+1}, k_{et+1})}{(1+g)(1+n)} \\ V_1^h(a_{et}, k_{et}) &= \frac{\beta^* V_1^h(a_{et+1}, k_{et+1})[1+\rho(k_{et})]}{(1+g)(1+n)} \end{aligned}$$

where

$$\begin{aligned} a_{et+1} &= \frac{(1+\rho(k_{et}))a_{et} + w(k_{et}) - c(a_{et}, k_{et})}{(1+g)(1+n)} \\ k_{et+1} &= \frac{f(k_{et}) + (1-\delta)k_{et} - C^m(k_{et})}{(1+g)(1+n)} \end{aligned}$$

# Recursive Competitive Equilibrium

- Firm

$$\begin{aligned} r(k_{et}) &= f'\left(k_e^f(k_{et})\right), \\ w(k_{et}) &= f\left(k_e^f(k_{et})\right) - r(k_{et})\, k_e^f(k_{et}) \end{aligned}$$

- Arbitrage condition

$$\rho(k_{et}) = r(k_{et}) - \delta$$

- Capital and Labor Market with initial condition

$$k_e^f(k_{et}) = k_{et} = a_{et}.$$

- the consistency of individual and aggregate decisions

$$C^m(k_{et}) = c(k_{et}, k_{et})$$

- Note that there exist $k_e^f(k_{et})$, $r(k_{et})$, $\rho(k_{et})$ and $w(k_{et})$ that satisfy the definition of RCE and

$$\begin{aligned} k_e^f(k_{et}) &= k_{et} \\ r(k_{et}) &= f'(k_{et}) \\ \rho(k_{et}) &= f'(k_{et}) - \delta, \\ w(k_{et}) &= f(k_{et}) - f'(k_{et})\, k_{et}. \end{aligned}$$

# Recursive Competitive Equilibrium

- From the definition of the recursive competitive equilibrium

$$
\begin{aligned}
a_{et+1} &= \frac{(1+r(k_{et})-\delta)k_{et}+w(k_{et})-c(k_{et},k_{et})}{(1+g)(1+n)} \\
&= \frac{(1+r(k_{et})-\delta)k_{et}+f(k_{et})-r(k_{et})k_{et}-c(k_{et},k_{et})}{(1+g)(1+n)} \\
&= \frac{f\left(k_e^f(k_{et})\right)+(1-\delta)k_{et}-c(k_{et},k_{et})}{(1+g)(1+n)} \\
&= \frac{f(k_{et})+(1-\delta)k_{et}-C^m(k_{et})}{(1+g)(1+n)} \\
&= k_{et+1}
\end{aligned}
$$

# Recursive Competitive Equilibrium

- Therefore, households' first order condition is

$$\tilde{u}'\left(C^m\left(k_{et}\right)\right) = \frac{\beta^* V_1^h\left(k_{et+1}, k_{et+1}\right)}{\left(1+g\right)\left(1+n\right)}$$

- Envelope theorem is

$$V_1^h\left(k_{et}, k_{et}\right) = \frac{\beta^* V_1^h\left(k_{et+1}, k_{et+1}\right)\left[f'\left(k_{et}\right)+1-\delta\right]}{\left(1+g\right)\left(1+n\right)}$$

# Recursive Competitive Equilibrium

- Define $V^{m\prime}(k_{et}) \equiv V_1^h(k_{et}, k_{et})$

$$
\begin{aligned}
\tilde{u}'(C^m(k_{et})) &= \frac{\beta^* V^{m\prime}(k_{et})}{(1+g)(1+n)} \\
V^{m\prime}(k_{et}) &= \frac{\beta^* V^{m\prime}(k_{et+1})\left[f'(k_{et}) + 1 - \delta\right]}{(1+g)(1+n)}
\end{aligned}
$$

- Remember that Social Planner Problem is summarized by

$$\tilde{u}'\left(C\left(k_{et}\right)\right) = \frac{\beta^* V'\left(k_{et+1}\right)}{\left(1+g\right)\left(1+n\right)}$$

$$V'\left(k_{et}\right) = \frac{\beta^* V'\left(k_{et+1}\right)\left[f'\left(k_{et}\right)+\left(1-\delta\right)\right]}{\left(1+g\right)\left(1+n\right)}$$

- We know that there exists the set of policy functions and the marginal value function $\left\{C\left(k_{et}\right), V'\left(k_{et}\right)\right\}$ that satisfies these equations exists and unique. It means that $\left\{C^m\left(k_{et}\right), V^{m\prime}\left(k_{et}\right)\right\}$ exists and

$$C^m\left(k_{et}\right) = C\left(k_{et}\right), V^{m\prime}\left(k_{et}\right) = V'\left(k_{et}\right).$$

- Given the existence of the aggregate consumption function $C^m(k_{et})$ and the price functions $\rho(k_{et})$ and $w(k_{et})$, there exists $\left\{c(a_{et}, k_{et}), V_1^h(a_{et}, k_{et})\right\}$ that satisfies the definition of RCE because these are the solutions to the standard Dynamic Programming given the price functions and the aggregate decisions.

# Recursive Competitive Equilibrium

- We have proved that there is a price mechanism that can reproduce social optimal allocation.
- Obviously, we are interested in a decentralized economy, but not a planner problem. But, because the social planner problem is easier to deal with, if you are interested in a perfectly competitive economy as a benchmark, you can analyze an optimal growth model.
- If you are interested in a distorted economy, you cannot use an optimal growth model. However, you can still use a recursive competitive equilibrium, though the proof of existence might be more difficult.
- Although I mainly discuss an optimal growth model in this lecture, you can apply a recursive competitive equilibrium to varieties of issues.

# Recursive Competitive Equilibrium

- **Assignment:** Consider the following social planner's problem.

$$\max_{\{C_t\}} \sum_{t=0}^{\infty} \beta^t NU\left(c_t\right),$$

$$s.t.\ K_{t+1} = F\left(K_t, N\right) + \left(1-\delta\right) K_t - c_t N,\ K_0 \textit{ is given.}$$

where $C_t$ is consumption per capita, $N$ is population, $K_t$ is capital stock, $\beta \in (0,1)$ is a discount factor, $\delta$ is the depreciation rate of capital stock. Population is assumed to be constant. Assume that the aggregate production function, $F(\cdot,\cdot)$, satisfies usual assumptions: it exhibits the constant return to scale, the first derivative is positive, the second is negative and it satisfies Inada condition. No population growth and no technological growth.

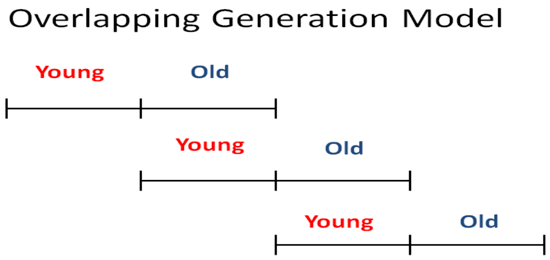
1. Define the recursive competitive equilibrium that can support the resource allocation of this problem.
2. Show that your equilibrium allocation is the same as the resource allocation of this problem.

# Overlapping Generation Model

- Recursive competitive equilibrium assumes that agents behave as if they never die. Some of you may think that this assumption is extreme.
- In order to understand how restrictive this assumption is and when this assumption is justified, this section briefly discusses about an overlapping generation model (OGM). The main difference from the recursive competitive equilibrium is that there is turnover in the population. New generation comes to society; old generation leaves. This model is especially useful when you are interested in interactions across generations.
- It is shown that (1) when we have turnover in the population, economy may not be Pareto optimal and (2) a recursive competitive equilibrium can be justified when parents care about their children very much.

- **Basic Model:**
  - For the simplest case, each agent lives in two periods; young and old.
  - In each period, new young enters the economy and old leaves. Therefore there is turnover in the population in each period.
  - The agents are identical in their generation, but differ across generation.

# Overlapping Generation Model



- **Household:** The agent with a constant risk aversion solves the following problem:

$$\max_{C_{yt}, C_{ot+1}} \left\{ \frac{c_{yt}^{(1-\theta)} - 1}{1-\theta} + \frac{1}{1+\varrho} \frac{c_{ot+1}^{(1-\theta)} - 1}{1-\theta} \right\}$$

$$s.t. s_t^n = w_t - c_{yt} \quad (10)$$

$$c_{ot+1} = \left(1 + \rho_{t+1}\right) s_t^n \quad (11)$$

where $s_t^n$ is net saving, $c_{yt}$ and $c_{ot}$ are consumption when he is young and old, respectively. When the agent is young, he earns wage $w_t$. He decides whether he consumes today or saves it for the next period. When he gets age, he retires and receives income from his saving. Since he does not have the next period, he consumes all his wealth when he is old.

- This model can be simplified by

$$\max_{s_t^n} \left\{ \frac{\left[w_t - s_t^n\right]^{(1-\theta)} - 1}{1-\theta} + \frac{1}{1+\varrho} \frac{\left[\left(1+\rho_{t+1}\right) s_t^n\right]^{(1-\theta)} - 1}{1-\theta} \right\}$$

- The first order condition is

$$\left[c_{yt}\right]^{-\theta} = \frac{\left(1+\rho_{t+1}\right)\left[c_{ot+1}\right]^{-\theta}}{1+\varrho}$$

Hence,

$$\frac{c_{ot+1}}{c_{yt}} = \left(\frac{1+\rho_{t+1}}{1+\varrho}\right)^{\frac{1}{\theta}} \tag{12}$$

- This is OGM version of the Euler equation. The agent compares the interest rate and the discount rate. When the interest rate is higher than the discount rate, he saves more and increases his consumption tomorrow. When the discount rate is larger, he increases his consumption today.

- Combing two budget constraints (10) and (11), I can derive the intertemporal budget constraint:

$$c_{yt} + \frac{c_{ot+1}}{1+\rho_{t+1}} = w_t. \tag{13}$$

It shows that the present value of the stream of lifetime consumption is equal to the present value of lifetime income, which, in this case, wages during young.

- Substituting the Euler equation (12) into (13),

$$\left[1 + \frac{\left(\frac{1+\rho_{t+1}}{1+\varrho}\right)^{\frac{1}{\theta}}}{1+\rho_{t+1}}\right] c_{yt} = w_t.$$

Therefore, consumption is linear in wages.

$$c_{yt} = \frac{(1+\varrho)^{\frac{1}{\theta}}}{(1+\varrho)^{\frac{1}{\theta}} + (1+\rho_{t+1})^{\frac{1-\theta}{\theta}}} w_t$$

- We can derive the saving rate, $s\left(\rho_{t+1}\right)$, as the function of the interest rate:

$$
\begin{aligned}
s\left(\rho_{t+1}\right) &= \frac{w_t - c_{yt}}{w_t} \\
&= \frac{\left(1+\rho_{t+1}\right)^{\frac{1-\theta}{\theta}}}{(1+\varrho)^{\frac{1}{\theta}} + \left(1+\rho_{t+1}\right)^{\frac{1-\theta}{\theta}}}
\end{aligned}
$$

As you can see, the saving rate can be an increasing or decreasing function of the interest rate:

$$
\begin{aligned}
s'\left(\rho_{t+1}\right) &> 0, \ if\ 1 > \theta \\
s'\left(\rho_{t+1}\right) &< 0, \ if\ 1 < \theta \\
s'\left(\rho_{t+1}\right) &= 0, \ if\ 1 = \theta
\end{aligned}
$$

- This is the result of a usual substitution effect and income effect. When the interest rate increases, the return to saving increases. Therefore the agent saves more (substitution effect). On the other hand an increase in the interest rate implies an increases in lifetime income. Therefore, the agent consumes more (income effect). As I discussed in the representative agent model, the parameter $\frac{1}{\theta}$ can be interpreted as the elasticity of substitution between consumption today and tomorrow. It measures the sensitivity of consumption to the change in price. When $\frac{1}{\theta}$ is larger than 1, $\theta$ is smaller than 1, the agent is sensitive to the change in the return to saving. Therefore, substitution effect dominates income effect. The saving rate is an increasing function of the interest rate.
- Using this saving rate, the net saving is

$$s_t^n = s\left(\rho_{t+1}\right) w_t$$

- *Firm:* Firms' maximization conditions are the same as before. The following profit maximization condition characterizes the firms' behavior:

$$\begin{aligned} r_t &= f'(k_{et}) \\ w_t &= \left[f(k_{et}) - f'(k_{et})\, k_{et}\right] T_t \end{aligned}$$

  As usual, I express the first order conditions by the efficiency unit term.

- *Intermediation:* The following arbitrage condition is also the same as before:

$$\rho_t = r_t - \delta$$

- *Labor Market Clearing Conditions:* Since only the young works in this economy, the demand for labor must be equal to the population of the young.

$$L_t = N_{yt}$$

where $N_{yt}$ is the population of the young.

- *Capital Market Clearing Conditions:* We assume that current net saving by young increases the asset at the next period. But when the agent gets olds, he consumes every asset he has, which reduces total asset in the economy in the next period. Therefore, the following condition must be satisfied.

$$A_{t+1} = A_t + s_t^n N_{yt} - s_{t-1}^n N_{yt-1}$$

Because this relationship must be satisfied for any $t$,

$$A_{t+1} = s_t^n N_{yt}$$

Since the demand for capital must be equal to the supply of asset,

$$K_{t+1} = A_{t+1} = s_t^n N_{yt}.$$

# Overlapping Generation Model

- We assume that the growth rate of productivity and population is $g$ and $n$, respectively:

$$\begin{aligned} T_{t+1} &= (1+g)\, T_t \\ N_{yt+1} &= (1+n)\, N_{yt} \end{aligned}$$

- Using efficiency unit, both market clearing conditions are summarized by

$$\begin{aligned} \frac{K_{t+1}}{T_{t+1} N_{yt+1}} \frac{T_{t+1}}{T_t} \frac{N_{yt+1}}{N_{yt}} T_t &= s_t^n \\ k_{et+1}\,(1+g)\,(1+n)\, T_t &= s_t^n . \end{aligned}$$

where $k_{et+1} = \frac{K_{t+1}}{T_{t+1} N_{yt+1}}$. Note that the denominator is not total population, but the number of young population because only young can work in this economy.

- **Equilibrium:** The market equilibrium consists of the sequence of $\left\{\left(s_t^n, \rho_{t+1}, w_t, r_t, k_t\right)\right\}_{t=0}^{\infty}$, which satisfies
- Consumer maximizes his utility:

$$\begin{aligned} s_t^n &= s\left(\rho_{t+1}\right) w_t \qquad (14) \\ \text{where } s\left(\rho_{t+1}\right) &= \frac{\left(1+\rho_{t+1}\right)^{\frac{1-\theta}{\theta}}}{\left(1+\varrho\right)^{\frac{1}{\theta}} + \left(1+\rho_{t+1}\right)^{\frac{1-\theta}{\theta}}} \end{aligned}$$

- Firm maximizes its profits:

$$\begin{aligned} r_t &= f'\left(k_{et}\right) \qquad (15) \\ w_t &= \left[f\left(k_{et}\right) - f'\left(k_{et}\right) k_{et}\right] T_t \qquad (16) \end{aligned}$$

- The arbitrage condition:

$$\rho_{t+1} = r_{t+1} - \delta \tag{17}$$

- Market clearing condition:

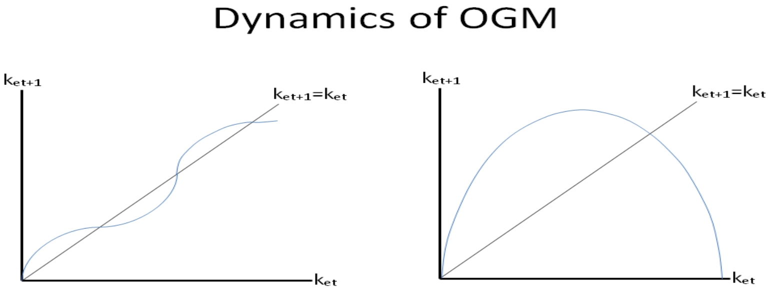
$$k_{et+1} (1+g) (1+n) T_t = s_t^n . \tag{18}$$

- Combining equilibrium conditions, we can derive

$$\begin{aligned} k_{et+1} &= \frac{s_t^n}{(1+g)(1+n)T_t} \\ &= \frac{s(\rho_{t+1})w_t}{(1+g)(1+n)T_t} \\ &= \frac{s(r_{t+1}-\delta)\left[f(k_{et})-f'(k_{et})k_{et}\right]T_t}{(1+g)(1+n)T_t} \\ &= \frac{s(f'(k_{et+1})-\delta)\left[f(k_{et})-f'(k_{et})k_{et}\right]}{(1+g)(1+n)} \end{aligned}$$

$$k_{et+1} = \frac{(f'(k_{et+1})+1-\delta)^{\frac{1-\theta}{\theta}}}{(1+\varrho)^{\frac{1}{\theta}}+(f'(k_{et+1})+1-\delta)^{\frac{1-\theta}{\theta}}}\frac{\left[f(k_{et})-f'(k_{et})k_{et}\right]}{(1+g)(1+n)} \tag{19}$$

- As you can see, this is a nonlinear first order difference equation. Hence, potentially, we can solve it given an initial condition $k_0$. Unfortunately, we cannot say much about the property of this dynamic equation. It is well known that the OGM can produce the variety of dynamics. It is possible to have multiple steady states in the OGM. In another case, we cannot determine the dynamics of OGM. Moreover, OGM can yield a chaotic fluctuation. The following figures are some examples.


Dynamics of OGM
ket+1
ket+1=ket
ket
ket+1
ket+1=ket
ket

- The dynamics of the OGM depends on the parameters of the production function and the utility function. This can be seen as the advantages and disadvantages of OGM. On one hand, it gives us possible explanations for several puzzling evidence including business cycle. On the other hand, it is difficult to tell where we stand and what would be a policy implication. Anything goes.

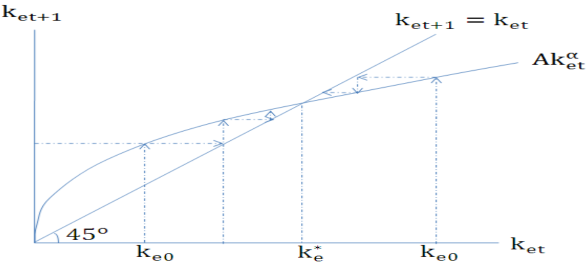
- Two additional assumptions simplify the dynamics of our solution. Assume that $\theta = 1$ and the production function is Cobb-Douglas $f(k) = k^{\alpha}$. Then equation (19) is

$$
\begin{aligned}
k_{et+1} &= \frac{(1-\alpha)(k_{et})^{\alpha}}{(2+\varrho)(1+g)(1+n)} \\
&\equiv A(k_{et})^{\alpha}
\end{aligned}
$$

where $A = \frac{(1-\alpha)}{(2+\varrho)(1+g)(1+n)}$

The Dynamics of OGM

when $\theta = 1$ and $f(k) = k^{\alpha}$



- In this case, there is a unique steady state and economy globally and monotonically converges to the steady state. Let me derive the steady state value of $k_{et}$. On the steady state $k_{et} = k_{et+1} = k_e^*$. Then

$$k_e^* = \left[ \frac{1-\alpha}{(2+\varrho)(1+g)(1+n)} \right]^{\frac{1}{1-\alpha}} \tag{20}$$

# Dynamic Inefficiency

- **Dynamic Inefficiency:** In order to discuss the inefficiency of the OGM, I examine the golden rule level of capital stock, $k_e^{GR}$, which maximizes consumption per unit of effective population when economy is stationary. Then I compare $k_e^{GR}$ and $k_e^*$.
- If $k_e^*$ is larger than $k_e^{GR}$, we can easily find new allocation in which everybody is better off. To see this, lower $k_e^*$ to $k_e^{GR}$ during period $t$. Since the agent can enjoy more consumption during period $t$, the agent is better off. From the next period, economy reaches $k_e^{GR}$. Since $k_e^{GR}$ maximizes consumption per unit of effective population, the following generations can enjoy higher consumption and they are better off. In other word, the steady state is not efficient.
- Remember that the optimal growth model always satisfy $k_e^* < k^{GR}$ because $\beta^* < 1$. As the household in OGM does not need to worry about infinite period later, this condition does not need to be satisfied.

# Dynamic Inefficiency

- Firstly, note that maximizing consumption per unit of effective population is the same as maximizing consumption per unit of effective labor:

$$\frac{C_t}{T_t\left[N_{yt}+N_{ot}\right]} = \frac{C_t}{T_t\left[N_{yt}+\frac{N_{yt}}{1+n}\right]}, \\ = \frac{1+n}{2+n}\frac{C_t}{T_t N_{yt}},$$

where $C_t = c_{yt}N_{yt} + c_{ot}N_{ot}$. Hence, I find the level of capital stock which maximizes consumption per workers given any level of technology.

# Dynamic Inefficiency

- Social planner faces resource constraint any time. The resource constraint is

$$K_{t+1} = F\left(K_t, T_t N_{yt}\right) - C_t + (1-\delta) K_t$$

This means

$$\begin{aligned} \frac{K_{t+1}}{T_{t+1}N_{yt+1}} \frac{T_{t+1}N_{yt+1}}{T_t N_{yt}} &= F\left(\frac{K_t}{T_t N_{yt}}, 1\right) - \frac{C_t}{T_t N_{yt}} + (1-\delta)\frac{K_t}{T_t N_{yt}} \\ k_{et+1}(1+g)(1+n) &= f(k_{et}) - c_{et} + (1-\delta) k_{et} \end{aligned}$$

where $c_{et} = \frac{C_t}{T_t N_{yt}}$ and $k_{et} = \frac{K_t}{T_t N_{yt}}$.

# Dynamic Inefficiency

- On the steady state $k_{et} = k_{et+1} = k_e^{**}$ and $c_{et} = c_e^{**}$. Hence,

$$c_e^{**} = f\left(k_e^{**}\right) + \left[(1-\delta) - (1+g)(1+n)\right] k_e^{**}$$

- Hence, the level of capital stock which maximizes consumption per workers given the level of productivity is

$$\frac{dc^{**}}{dk^{**}} = f'\left(k_e^{GR}\right) - \left[(1+g)(1+n) - (1-\delta)\right] = 0$$

Note that this golden rule is the same as the golden rule in the representative agent model.

# Dynamic Inefficiency

- Note that

$$\begin{aligned} \rho^* &= r^* - \delta \\ f'(k_e^*) &= r^* = \rho^* + \delta \end{aligned}$$

- Hence

$$\begin{aligned} k_e^{GR} &< k_e^* \ \textit{iff} \ (1+g)(1+n) - (1-\delta) > \rho^* + \delta \\ k_e^{GR} &< k_e^* \ \textit{iff} \ (1+g)(1+n) > 1 + \rho^* \end{aligned}$$

  - To maintain $k_e^{GR} < k_e^*$, the growth rate of technology and population growth must be large.
  - This is the condition that government debt per nominal GDP never explore as far as primary balance per nominal GDP does not explore.

- Assume that $f(k) = k^{\alpha}$, the golden rule level of capital stock per unit of effective labor is

$$\alpha\left(k_e^{GR}\right)^{\alpha-1} = (1+g)(1+n) - (1-\delta)$$

$$k_e^{GR} = \left[\frac{\alpha}{(1+g)(1+n) - (1-\delta)}\right]^{\frac{1}{1-\alpha}} \tag{21}$$

# Dynamic Inefficiency

- Because

$$k_e^* = \left[ \frac{1-\alpha}{(2+\varrho)(1+g)(1+n)} \right]^{\frac{1}{1-\alpha}}$$

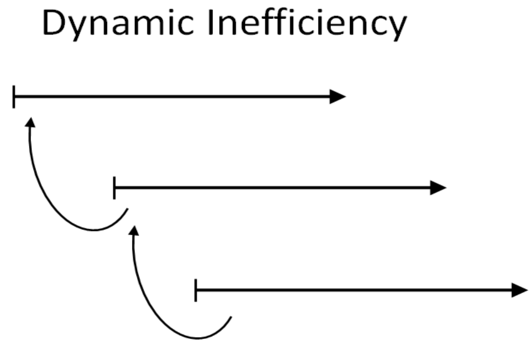
I get

$$k_e^{GR} < k_e^*, \ iff \ \frac{\alpha}{(1+g)(1+n)-(1-\delta)} < \frac{1-\alpha}{(2+\varrho)(1+g)(1+n)}.$$

# Dynamic Inefficiency

- Hence, there exists parameter values with which the steady state is not Pareto optimal. This result is contrasted with the one of the representative agent model in which the allocation is always Pareto optimal. This result sounds surprising since market is competitive and no externality in the overlapping generation model. The main reason is that we have the infinite number of agents in our economy. The social planner can transfer resources from young to old without a market. Obviously, the current old is better off and the current young is worse off. However, the social planner can compensate the current young when he becomes old by transferring resources from the next generation. Since we have the infinite number of generations, nobody may be worse off. This is the reason for inefficiency. This is called the dynamic inefficiency.

# Dynamic Inefficiency



- **Assignment:** A household with a constant risk aversion solves the following problem:

$$\max_{c_{yt}, c_{ot+1}} \left\{ \log c_{yt} + \frac{1}{1+\varrho} \log c_{ot+1} \right\}$$

$$s.t. s_t^n = (1-\tau) w_t - c_{yt}, \; c_{ot+1} = \left(1 + \rho_{t+1}\right) s_t^n + B_{t+1}$$

Government levies an income tax, $\tau$ and uses the proceeds to pay benefits to old individual by $B_t$. Suppose that Government balances budget constrained: $B_t = (1+n) \tau w_t$. Assume that production function is $K_t^{\alpha} \left(T_t L_t\right)^{(1-\alpha)}$ and $\delta = 1$. Maintain other assumptions in our lecture note.

1. Derive the difference equation of $k_{et} = \frac{K_t}{TN_{yt}}$. How is the equation influenced by $\tau$?
2. Derive the steady state level of capital stock per unit of effective workers, $k_e^*$. How is it influenced by $\tau$?
3. Compare the derived $k_e^*$ and $k^{GR}$. Discuss how $\tau$ affects the welfare.

# Altruism

- The above reasoning suggests that if one generation cares about the next generation, we may be able to recover efficiency.
- This motivates us to modify the model to include altruism.

# Altruism

- Suppose that the agents cares about their children. More concretely, the young during period $t$ maximizes the following utility function:

$$U_t = u(C_{yt}) + \frac{1}{1+\varrho}\left[u(c_{ot+1}) + \frac{1+n}{1+\varphi}U_{t+1}\right],$$

where $\varphi$ is the measure of selfishness. If $\varphi = 0$, parents treat their children like themselves, but if $\varphi = \infty$, parents do not care about their children at all. The variable $U_t$ is the total sum of discounted utility of the young during period $t$. During period $t+1$, an agent expects to have $1+n$ children. Since the parents are selfish in that they care about themselves more than their children, the benefits from their children are discounted by $\frac{1}{1+\varphi}$. Hence their utility from their children is $\frac{1+n}{1+\varphi}U_{t+1}$.

# Altruism

- Assume that
$$0 = \lim_{s\to\infty} \beta^s U_{t+s}.$$
where $\beta = \frac{1+n}{(1+\varrho)(1+\varphi)}$. Then utility of generation $t$ is

$$\begin{aligned}
U_t &= u(C_{yt}) + \frac{1}{1+\varrho} u(c_{ot+1}) + \beta U_{t+1} \\
&= u(C_{yt}) + \frac{1}{1+\varrho} u(c_{ot+1}) + \beta \left[ \begin{array}{c} u(C_{yt+1}) + \frac{1}{1+\varrho} u(c_{ot+2}) \\ +\beta U_{t+2} \end{array} \right] \\
&= u(C_{yt}) + \frac{1}{1+\varrho} u(c_{ot+1}) \\
&\quad +\beta \left[ u(C_{yt+1}) + \frac{1}{1+\varrho} u(c_{ot+2}) \right] + \beta^2 U_{t+2} \\
&= \sum_{s=0}^{\infty} \beta^s \left[ u(c_{yt+s}) + \frac{1}{1+\varrho} u(c_{ot+s+1}) \right] + \lim_{s\to\infty} \beta^s U_{t+s}
\end{aligned}$$

# Altruism

- $U_t$

$$U_t = \sum_{s=0}^{\infty} \beta^s \left[ u\left(c_{yt+s}\right) + \frac{1}{1+\varrho} u\left(c_{ot+s+1}\right) \right]$$

- Assume that parents can transfer their income to their children. Then the budget constraint of each generation must change:

$$c_{yt} + s_t = w_t + m_t \quad (22)$$

$$c_{ot+1} + (1+n)\, m_{t+1} = \left(1 + \rho_{t+1}\right) s_t \quad (23)$$

where $m_t$ is the transfer from the parents to their children. Hence,

$$\begin{aligned} (1+n)\, m_{t+1} &= \left(1+\rho_{t+1}\right) \left[ w_t + m_t - c_{yt} \right] - c_{ot+1} \\ m_{t+1} &= \frac{1+\rho_{t+1}}{1+n} \left[ w_t + m_t - c_{yt} \right] - \frac{c_{ot+1}}{1+n} \end{aligned}$$

# Altruism

- This problem is equivalent to

$$\max_{\{C_{yt+s},C_{ot+s+1}\}} \sum_{s=0}^{\infty} \beta^s \left[ u\left(c_{yt+s}\right) + \frac{1}{1+\varrho} u\left(c_{ot+s+1}\right) \right]$$

$$s.t\ m_{t+1} = \frac{1+\rho_{t+1}}{1+n} \left[ w_t + m_t - c_{yt} \right] - \frac{c_{ot+1}}{1+n}$$

where $\beta = \frac{1+n}{(1+\varrho)(1+\varphi)} < 1$.

- This is essentially the problem of representative agent. In the representative agent model, the market economy is Pareto optimal. (Remember that a social planner model can be attained by RCE.) The overlapping generation model with altruism can also attain Pareto optimal. Because parents care about children's utility and budget constraint is connected by bequests, any transfer from young to old cannot improve old's utility.

# Continuous Time Problem

- We discusses how the discrete dynamic programming can be applied for the optimal growth model.
- However, this is not the only tool to analyze a dynamic optimization problem. Depending on the model, it is easier to analyze continuous time model.
- In particular, it is easier to analyze a transition dynamics using continuous time model.
- We will discuss the continuous time model as a limit of the discrete time model.

# Continuous Time Problem

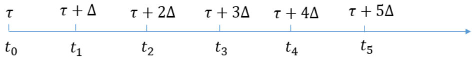
- In the previous model, time goes like $t$, $t+1$, $t+2$.
- In this section, I assume that time goes like $t$, $t+\Delta$, $t+2\Delta$....Then I describe the continuous time model by taking a limit of $\Delta$: $\Delta \to 0$.
- I assume that one period return between $t$ and $t+\Delta$ is constant, $\Delta r(X_t, S_t)$. Therefore, the sum of the returns is

$$\Delta r(X_\tau, S_\tau) + \beta \Delta r(X_{\tau+\Delta}, S_{\tau+\Delta}) + \beta^2 \Delta r(X_{\tau+2\Delta}, S_{\tau+2\Delta}) + ...,$$

  where $\tau$ is an initial period. We can define the time that an agent can make the $j$th decision by $t_j = \tau + j\Delta$. Hence, $j = \frac{t_j - \tau}{\Delta}$.
- Therefore, the above sum can be rewritten as

$$\Delta r(X_\tau, S_\tau) + \beta^{\frac{t_1-\tau}{\Delta}} \Delta r(X_{t_1}, S_{t_1}) + \beta^{\frac{t_2-\tau}{\Delta}} \Delta r(X_{t_2}, S_{t_2}) ...$$

# Continuous Time Problem



- Assume that

$$\beta = \frac{1}{1+\Delta\varrho},$$

where $\varrho$ is the discount rate.

- Then the continuous version of the original model can be expressed as

$$\begin{aligned} U(S_\tau) &= \max_{\{X_t\}} \left\{ \lim_{\Delta\to 0} \sum_{j=0}^{\infty} \left( \frac{1}{1+\Delta\varrho} \right)^{\frac{(t_j-\tau)}{\Delta}} \Delta r\left(X_{t_j}, S_{t_j}\right) \right\}, \\ s.t.\ S_{t+\Delta} &= G(X_t, S_t) = \Delta G^c(X_t, S_t) + S_t, \\ & S_\tau \text{ is given.} \end{aligned}$$

# Continuous Time Problem

- Since

$$e^{-\varrho t} = \lim_{\Delta \to 0} \left[ \frac{1}{1 + \Delta \varrho} \right]^{\frac{t}{\Delta}},$$

- the continuous version of the previous model is written as

$$\begin{aligned} U(S_\tau) &= \max_{X_t} \int_\tau^\infty e^{-\varrho(t-\tau)} r(X_t, S_t)\, dt \\ s.t.\ \dot{S}_t &= G^c(X_t, S_t) \\ & S_\tau \textit{ is given.} \end{aligned}$$

# Continuous Time Problem

- I would like to derive the continuous version of the Bellman equation as the limit of the previous discrete model. Since $\beta = \frac{1}{1+\Delta\rho}$

$$\begin{aligned} V(S_t) &= \max_{X_t} \left\{ \Delta r(X_t, S_t) + \frac{1}{1+\Delta\varrho} V(S_{t+\Delta}) \right\}, \\ S_{t+\Delta} - S_t &= \Delta G^c(X_t, S_t). \end{aligned}$$

- We can modify the value function as follows.

$$\begin{aligned} (1+\Delta\varrho) V(S_t) &= \max_{X_t} \left\{ (1+\Delta\varrho) \Delta r(X_t, S_t) + V(S_{t+\Delta}) \right\} \\ \Delta\varrho V(S_t) &= \max_{X_t} \left\{ \begin{array}{c} \Delta r(X_t, S_t) + \Delta^2 \varrho r(X_t, S_t) \\ + \left[ V(S_{t+\Delta}) - V(S_t) \right] \end{array} \right\} \\ \varrho V(S_t) &= \max_{X_t} \left\{ \begin{array}{c} r(X_t, S_t) + \Delta\varrho r(X_t, S_t) \\ + \frac{V(S_{t+\Delta}) - V(S_t)}{\Delta} \end{array} \right\} \end{aligned}$$

# Continuous Time Problem

- When $\Delta$ goes 0, the continuous version of the Bellman equation is derived as follows.

$$\begin{aligned} \varrho V(S_t) &= \max_{X_t} \left\{ r(X_t, S_t) + V'(S_t)\dot{S}_t \right\} \\ &= \max_{X_t} \left\{ r(X_t, S_t) + V'(S_t) G^c(X_t, S_t) \right\} \end{aligned}$$

- Bellman Equation:

$$\varrho V(S_t) = \max_{X_t} \left\{ r(X_t, S_t) + V'(S_t) G^c(X_t, S_t) \right\}$$

# Continuous Time Problem

- The first order condition of this Bellman equation is

$$0 = r_1\left(x\left(S_t\right), S_t\right) + V'\left(S_t\right) G_1^c\left(x\left(S_t\right), S_t\right). \tag{24}$$

- Substitute the policy function into the Bellman equation,

$$\varrho V\left(S_t\right) = r\left(x\left(S_t\right), S_t\right) + V'\left(S_t\right) G^c\left(x\left(S_t\right), S_t\right). \tag{25}$$

Again, the first order condition (24) and the Bellman equation (25) solves the value function $V(\cdot)$ and $x(\cdot)$. We can apply the Guess and verified method to solve two equations.

- *Example:* Consider the following investment problem:

$$\max_{L_t, I_t} \int_0^{\infty} \left[ F\left(K_t, L_t\right) - wL_t - p\left(C\left(I_t, K_t\right) + I_t\right)\right] e^{-\rho t} dt$$

$$s.t.\ \dot{K}_t = I_t - \delta K_t,\ K_0 \text{ is given}$$

where $F(K, L)$ and $C(I, K)$ are constant returns to scale in each factors.

- The Bellman Equation of this problem can be written as

$$\rho V(K) = \max_{I,L} \left\{ F(K, L) - wL - p(C(I, K) + I) + V'(K)(I - \delta K) \right\}$$

# Guess and Verify Method

- Guess $V(K) = QK$.

$$\begin{aligned} & \max_{I,L} \left\{ F(K,L) - wL - p\left(C(I,K) + I\right) + Q\left(I - \delta K\right) \right\} \\ = & \max_{I,L} \left\{ \begin{array}{c} F\left(1, \frac{L}{K}\right) - w\frac{L}{K} \\ -p\left(C\left(\frac{I}{K}, 1\right) + \frac{I}{K}\right) + Q\frac{I-\delta K}{K} \end{array} \right\} K \\ = & \max_{g,l} \left\{ \phi(l) - wl - p\left(c(g) + g\right) + Q\left[g - \delta\right] \right\} K \end{aligned}$$

where $\phi(l) = F(1,l)$, $c(g) = C(g,1)$, $l = \frac{L}{K}$ and $g = \frac{I}{K}$. If there exists $Q$ that satisfies

$$\rho Q = \max_{g,l} \left\{ \phi(l) - wl - p\left(c(g) + g\right) + Q\left[g - \delta\right] \right\}.$$

Then our guess is verified. Since the value function is unique, the linear must be the property of the value function.

# Guess and Verify Method

- Note that

$$
\begin{aligned}
Q &= p\left[1 + c'(g)\right] \\
g &= \psi\left[Q^{tobin} - 1\right], \quad Q^{tobin} = \frac{Q}{p}
\end{aligned}
$$

- This is called the $Q$ theory of investment. It says that the firm's growth rate is determined by $Q^{tobin}$. Note that $Q^{tobin}$ can be estimated by

$$
Q^{tobin} = \frac{V(K)}{pK}.
$$

This is the market value of a firm relative to the replacement cost of capital.

# Guess and Verify Method

- Assume that $c(g) = \frac{1}{2A}(g - B)^2$ and $c'(g) = \frac{(g-B)}{A}$. Hence

$$
\begin{aligned}
\frac{(g-B)}{A} &= \left[Q^{tobin} - 1\right] \\
g &= B + A\left[Q^{tobin} - 1\right] \\
&= B^* + AQ^{tobin} + \varepsilon
\end{aligned}
$$

where $B^* + \varepsilon = B - A$. Empirical researchers often estimate this regression.

# Continuous Time Problem

- **Assignment:** Consider the following household problem:

$$\max_{c_t} \int_0^{\infty} \ln c_t e^{-\varrho t} dt,$$
$$s.t. \ \dot{a}_t = \rho a_t - c_t, a_0 \text{ is given}$$

where $a_t$ is asset, $w_t$ is wage and $c_t$ is consumption.

1. Define the Bellman equation.
2. Derive a Policy Function.

# Continuous Time Problem

- The envelope theorem can be also applied to the Bellman equation.

$$\varrho V'(S_t) = r_2(X_t, S_t) + V''(S_t) G^c(X_t, S_t) + V'(S_t) G_2^c(X_t, S_t).$$

  - Note that $V''(S_t) G^c(X_t, S_t) = V''(S_t) \dot{S}_t = \frac{dV'(S_t)}{dt}$. Hence the envelope theorem can be rewritten as

$$\frac{dV'(S_t)}{dt} = \varrho V'(S_t) - \left[r_2(X_t, S_t) + V'(S_t) G_2^c(X_t, S_t)\right]. \quad (26)$$

  - This equation describes the dynamics of the marginal value of state variable $V'(S_t)$.

# Continuous Time Problem

- **Optimal Control:** Defining the costate variable, $\lambda_t$, is equal to the marginal value of state variables,

$$\lambda_t \equiv V'(S_t),$$

we can derive the result of alternative method to solve dynamic optimization, optimal control.

- Using $\lambda_t$, the first order condition (24) and the envelope theorem (26) and the transition equation can be rewritten as

$$0 = r_1(x^o(S_t, \lambda_t), S_t) + \lambda_t G_1^c(x^o(S_t, \lambda_t), S_t), \quad (27)$$

$$\dot{\lambda}_t = \varrho \lambda_t - [r_2(x^o(S_t, \lambda_t), S_t) + \lambda_t G_2^c(x^o(S_t, \lambda_t), S_t)], \quad (28)$$

$$\dot{S}_t = G^c(x^o(S_t, \lambda_t), S_t), \ S_\tau \text{ is given.} \quad (29)$$

where $x^o(S_t, V'(S_t)) = x(S_t)$.

- The first order condition (27) determines a policy function $x^o(S_t, \lambda_t)$. Given the policy function $x^o(S_t, \lambda_t)$, equations (28) and (29) determine the path of the costate variable, $\lambda_t$ and the state variable, $S_t$.
- Note that since we have one boundary condition, $S_\tau$, we need an additional boundary condition to solve the equations (28) and (29). For this purposes, we need the following transversality condition:

$$\lim_{t\to\infty} e^{-\varrho(t-\tau)} V'(S_t) S_t = \lim_{t\to\infty} e^{-\varrho(t-\tau)} \lambda_t S_t = 0. \tag{30}$$

- Similar to the discrete time problem, given usual technical assumptions, concavity etc, it is shown that equations (27), (28), (29) and (30) are sufficient conditions for the original problem.

# Continuous Time Problem

- The above conditions are nicely summarized by defining Hamiltonian $H\left(X_t, S_t, \lambda_t\right)$:

$$H\left(X_t, S_t, \lambda_t\right) = r\left(X_t, S_t\right) + \lambda_t G^c\left(X_t, S_t\right).$$

- Then above conditions are expressed as

$$\begin{aligned}
0 &= H_X\left(X_t, S_t, \lambda_t\right) \\
\dot{\lambda}_t &= \varrho\lambda_t - H_S\left(X_t, S_t, \lambda_t\right) \\
\dot{S}_t &= H_\lambda\left(X_t, S_t, \lambda_t\right),\ S_\tau \text{ is given.} \\
0 &= \lim_{t\to\infty} e^{-\varrho(t-\tau)}\lambda_t S_t
\end{aligned}$$

These are the useful conditions for analyzing the continuous version of the dynamic optimization problem.

- Hamiltonian can be nicely interpreted. When we choose a dynamic optimization problem, we know that current decision affects not only the current return, but also the future returns. The second term of Hamiltonian summarizes the impact on the future returns. Remember, $\lambda_t = V'(S_t)$. That is, the costate variable can be interpreted as the marginal impact of the state variable on the present value of the discounted future returns. Since $\dot{S}_t = G^c(X_t, S_t)$, the second term is interpreted as the impact of a change in $S_t$ on the future returns. Therefore, Hamiltonian summarizes the important trade off of the dynamic optimization problem: the impact on the current return and the future returns. The first order condition, $0 = H_X$, implies that the agent chooses the control variable $X_t$ so that he maximizes Hamiltonian.

- *Example:* Consider the following investment problem:

$$\max_{L_t, I_t} \int_{\tau}^{\infty} \left[ F\left(K_t, L_t\right) - wL_t - p\left(C\left(I_t, K_t\right) + I_t\right)\right] e^{-\rho t} dt$$
$$s.t.\ \dot{K}_t = I_t - \delta K_t,\ \ K_\tau \text{ is given}$$

- Hamiltonian

$$\begin{aligned} & H\left(L_t, I_t, K_t, q_t\right) \\ = \ & F\left(K_t, L_t\right) - wL_t - p\left(C\left(I_t, K_t\right) + I_t\right) + q_t\left[I_t - \delta K_t\right]. \end{aligned}$$

# Continuous Time Problem

- Derive the necessary conditions and transversality condition for this problem.

$$
\begin{aligned}
0 &= H_L(L_t, I_t, K_t, q_t) = F_L(K_t, L_t) - w \\
0 &= H_I(L_t, I_t, K_t, q_t) = q_t - p(1 + C_I(I_t, K_t)) \\
\dot{q}_t &= \rho q_t - H_K(L_t, I_t, K_t, q_t) \\
&= \rho q_t - [F_K(K_t, L_t) - pC_K(I_t, K_t) - \delta q_t] \\
\dot{K}_t &= H_q(L_t, I_t, K_t, q_t) = I_t - \delta K_t, \ \ K_\tau \text{ is given.} \\
0 &= \lim_{t \to \infty} e^{-\rho(t-\tau)} q_t K_t
\end{aligned}
$$

# Continuous Time Problem

- Marginal Productivity of Labor is equal to wage.

$$F_L(K_t, L_t) = w$$

- The value of an additional unit of installed capital, $q_t$, is the marginal cost, which is equal to the price of investment good plus the marginal adjustment cost.

$$q_t = [1 + C_I(I_t, K_t)]\, p$$

  - Note that if $C_I(I_t, K_t) = 0$, $q_t = p$.

# Continuous Time Problem

- The interpretation of $q_t$: Define $\pi_K = F_K(K, L) - pC_K(I, K)$ is the marginal cash flow attributed to a unit of capital.

$$
\begin{aligned}
\dot{q}_t &= \rho q_t - \pi_{K,t} + \delta q_t \\
\rho q_t &= \pi_K - \delta q_t + \dot{q}_t \\
\rho &= \frac{\pi_K - \delta q_t + \dot{q}_t}{q_t}
\end{aligned}
$$

  - The shadow price of capital, $q_t$, is the price at which a marginal unit of installed capital could be bought or sold.

# Continuous Time Problem

- Note that if $C_K(I, K) = 0$, $\pi_K = F_K(K, L)$. Then, we can define the user cost (=rental price) $r_t = \pi_{K,t}$. Hence,

$$\begin{aligned} \rho q_t &= r_t - \delta q_t + \dot{q}_t \\ &\approx r_t - \delta q_t + q_{t+1} - q_t \\ (1+\rho) q_t &\approx r_t + (1-\delta) q_{t+1} \\ r_t &= \pi_{K,t} = F_K(K, L) \end{aligned}$$

  - This is the dynamics of $q_t$ which I have derived in Macroeconomic Analysis.

- Suppose that not only $C_K(I,K)=0$, but also $C_I(I,K)=0$. Then $q_t = p_t$. Hence

$$\rho = \frac{r_t - \delta p_t + \dot{p}_t}{p_t}$$
$$r_t = \pi_{K,t} = F_K(K,L)$$

  - Furthermore, Suppose $p_t = 1$ over time.

$$\rho = r_t - \delta$$

  - This is the arbitrage condition in Macroeconomic Analysis.

- In order to obtain an alternative interpretation of $q_t$, we need to know how to solve a differential equation.

# Continuous Time Problem

## Lemma

*Suppose that $x_t$ follows a differential equation:*

$$\frac{dx_t}{dt} = a_t + b_t x_t$$

1. *Suppose that $x_\tau$ is given. Then the solution to this differential equation is*

$$x_T = e^{\int_\tau^T b_s ds} \left[ x_\tau + \int_\tau^T a_t e^{-\int_\tau^t b_s ds} dt \right]$$

2. *Suppose that $0 = \lim_{T \to \infty} x_T e^{-\int_t^T b_s ds}$. Then the solution to this differential equation is*

$$x_\tau = -\int_\tau^\infty a_t e^{-\int_\tau^t b_s ds} dt$$

- **Proof:** Note that the differential equation can be rewritten as

$$
\begin{aligned}
a_t e^{-\int_\tau^t b_s ds} &= \frac{dx_t}{dt} e^{-\int_\tau^t b_s ds} - b_t x_t e^{-\int_\tau^t b_s ds} \\
&= \frac{d\left[x_t e^{-\int_\tau^t b_s ds}\right]}{dt} \\
\int_\tau^T a_t e^{-\int_\tau^t b_s ds} dt &= \int_\tau^T \frac{d\left[x_t e^{-\int_\tau^t b_s ds}\right]}{dt} dt \\
&= x_T e^{-\int_\tau^T b_s ds} - x_\tau
\end{aligned}
$$

# Continuous Time Problem

- **Proof:**
  - Suppose that $x_\tau$ is given.

$$x_T = e^{\int_\tau^T b_s ds}\left[x_\tau + \int_\tau^T a_t e^{-\int_\tau^t b_s ds} dt\right]$$

  - Suppose that $0 = \lim_{T\to\infty} x_T e^{-\int_t^T b_s ds}$.

$$x_\tau = -\int_\tau^\infty a_t e^{-\int_\tau^t b_s ds} dt$$

# Continuous Time Problem

- Alternative interpretation of $q_t$: Applying the lemma in this context.

$$
\begin{aligned}
\dot{q}_t &= \rho q_t - \pi_{K,t} + \delta q_t \\
\dot{q}_t - (\rho + \delta) q_t &= -\pi_{K,t} \\
\dot{q}_t e^{-(\rho+\delta)(t-\tau)} - (\rho + \delta) q_t e^{-(\rho+\delta)(t-\tau)} &= -\pi_{K,t} e^{-(\rho+\delta)(t-\tau)} \\
\frac{d q_t e^{-(\rho+\delta)(t-\tau)}}{dt} &= -\pi_{K,t} e^{-(\rho+\delta)(t-\tau)} \\
\left[ q_t e^{-(\rho+\delta)(t-\tau)} \right]_\tau^\infty &= -\int_\tau^\infty \pi_{K,t} e^{-(\rho+\delta)(t-\tau)} dt \\
\lim_{t\to\infty} q_t e^{-(\rho+\delta)(t-\tau)} - q_\tau &= -\int_\tau^\infty \pi_{K,t} e^{-(\rho+\delta)(t-\tau)} dt
\end{aligned}
$$

# Continuous Time Problem

- Hence,

$$q_\tau = \int_\tau^\infty \pi_{K,t} e^{-(\rho+\delta)(t-\tau)} dt + \lim_{t\to\infty} q_t e^{-(\rho+\delta)(t-\tau)}$$

- Note that TVC implies $0 = \lim_{t\to\infty} e^{-\rho(t-\tau)} q_t K_t$. Suppose that there exists $B$ such that $K_t \leq B < \infty$. Then $0 = \lim_{t\to\infty} q_t e^{-(\rho+\delta)(t-\tau)}$. Hence

$$q_\tau = \int_\tau^\infty \pi_{K,t} e^{-(\rho+\delta)(t-\tau)} dt$$

  - The shadow price of capital, $q_t$, is equal to the present discounted value of the stream of marginal cash flow attributed to a unit of capital installed at time t.

# Continuous Time Problem

- **Assignment:** Consider the following household problem:

$$\max_{c_t} \int_0^{\infty} \ln c_t e^{-\varrho t} dt,$$
$$s.t. \ \ \dot{a}_t = \rho a_t + w_t - c_t, a_0 \ is \ given$$

where $a_t$ is asset, $w_t$ is wage and $c_t$ is consumption.

1. Define Hamiltonian.
2. Derive the first order conditions.

# Continuous Time Problem

- **Assignment:** Consider the following budget constraint

$$\dot{a}_t = \rho_t a_t + w_t - \tau_t - c_t, \; a_0 \textit{ is given}$$

where $c_t$ is consumption, $a_t$ is asset, $w_t$ is wage payment, $\tau_t$ is tax payment and $\rho_t$ is interest rate. Suppose that the following condition is satisfied

$$0 = \lim_{T \to \infty} a_T e^{-\int_0^T \rho_s ds}.$$

This is called no Ponzi game condition, which means that debt cannot increase faster than the interest rate. Derive the following intertemporal budget constraint:

$$\int_0^\infty c_t e^{-\int_0^t \rho_s ds} dt = h_0 + a_0 - \int_0^\infty \tau_t e^{-\int_0^t \rho_s ds} dt,$$

where $h_0 = \int_0^\infty w_t e^{-\int_0^t \rho_s ds} dt$ is human wealth at date 0.

- **Assignment:** Consider the following government budget constraint

$$\dot{b}_t = g_t - \tau_t + \rho_t b_t, \ \ b_0 \text{ is given}$$

where $b_t$ is government bond, $\tau_t$ is tax revenue, $\rho_t$ is interest rate and $g_t$ is government expenditure. Suppose that the following condition is satisfied

$$0 = \lim_{T\to\infty} b_T e^{-\int_0^T \rho_s ds}.$$

This is called no Ponzi game condition for government. Derive the following intertemporal budget constraint:

$$b_0 + \int_0^\infty g_t e^{-\int_0^t \rho_s ds} dt = \int_0^\infty \tau_t e^{-\int_0^t \rho_s ds} dt$$

- **Assignment:** Using results in the previous two assignments and the following capital market clearing condition,

$$a_t = k_t + b_t, \ \ for \ any \ \ t$$

Derive the following equation

$$\int_0^\infty c_t e^{-\int_0^t \rho_s ds} dt = h_0 + k_0 - \int_0^\infty g_t e^{-\int_0^t \rho_s ds} dt$$

Note that this budget constraint does not include neither debt, $b_t$, and tax, $\tau_t$. Therefore, the method of financing does not change the budget constraint of the representative consumer. This is called Ricardian Equivalence

# Continuous Time Growth Model

- In this section I apply optimal control to the neoclassical growth model and analyze the neoclassical growth model. Consider the following problem.

$$\max_{c_t} \int_0^\infty e^{-\varrho t} N_t U(c_t)\, dt$$

$$\begin{aligned} \dot{K}_t &= F(K_t, T_t N_t) - \delta K_t - c_t N_t, \\ \dot{T}_t &= g T_t, \\ \dot{N}_t &= n N_t, \end{aligned}$$

where $K_0$, $T_0$ and $N_0$ are given.

- Note that

$$g_{k_{et}} = g_{\frac{K}{TN}} = g_K - (g_T + g_N)$$

Hence

$$\begin{aligned}
\frac{\dot{k}_{et}}{k_{et}} &= \frac{F(K_t, T_t N_t) - \delta K_t - c_t N_t}{K_t} - \left( \frac{\dot{T}_t}{T_t} + \frac{\dot{N}_t}{N_t} \right), \\
&= \frac{[F(k_{et}, 1) - \delta k_{et} - c_{et}] \, T_t N_t}{K_t} - (g + n), \\
&= \frac{[f(k_{et}) - \delta k_{et} - c_{et}]}{k_{et}} - (g + n), \\
\dot{k}_{et} &= f(k_{et}) - c_{et} - (g + n + \delta) k_{et},
\end{aligned}$$

where $f(k_t) = F(k_t, 1)$. From the second equation to the third, I use the assumption on the production function, the constant return to scale.

# Continuous Time Growth Model

- **Assignment:**

1. Prove that if $\dot{X}_t = gX_t$, $X_t = X_0 e^{gt}$.
2. Suppose that $U(c_{et} T_t) = \frac{(c_{et} T_t)^{(1-\theta)} - 1}{1-\theta}$, $\theta \geq 0$. Show that the original problem is equivalent to

$$\max_{c_{et}} \int_0^\infty e^{-\varrho^* t} \tilde{u}(c_{et})\, dt,$$

$$\dot{k}_{et} = f(k_{et}) - c_{et} - (g + n + \delta) k_{et}, \; k_{e0} \text{ is given}$$

where $\tilde{u}(c_{et}) = \frac{(c_{et})^{(1-\theta)}}{1-\theta}$ if $0 \leq \theta \neq 1$ and $\tilde{u}(c_{et}) = \ln c_{et}$ if $\theta = 1$, and $\varrho^* = \varrho - n - (1-\theta) g$. We assume that $\varrho^* > 0$. [Hint: the above statement implies $N_t = N_0 e^{nt}$ and $T_t = T_0 e^{gt}$. ]

- Define Hamiltonian of this problem:

$$H\left(c_{et}, k_t, \lambda_t\right) = \frac{\left(c_{et}\right)^{(1-\theta)}}{1-\theta} + \lambda_t\left[f\left(k_{et}\right) - c_{et} - \left(g+n+\delta\right)k_{et}\right]$$

- The first order conditions are

$$
\begin{aligned}
\lambda_t &= c_{et}^{-\theta} \\
\dot{\lambda}_t &= \varrho^* \lambda_t - \lambda_t \left[ f'\left(k_{et}\right) - \left(g + n + \delta\right) \right],\ 0 = \lim_{t \to \infty} \lambda_t k_{et} e^{-\rho t} \\
\dot{k}_{et} &= f\left(k_{et}\right) - c_{et} - \left(g + n + \delta\right) k_{et},\ \ k_{e0} \textit{ is given}
\end{aligned}
$$

- Note that

$$g_\lambda = g_{c_e^{-\theta}} = -\theta g_{c_e}$$

- Hence

$$\begin{aligned} -\theta \frac{\dot{c}_{et}}{c_{et}} &= \frac{\dot{\lambda}_t}{\lambda_t} = \varrho^* - \left[ f'(k_{et}) - (g+n+\delta) \right] \\ &= [\varrho - n - (1-\theta) g] - \left[ f'(k_{et}) - (g+n+\delta) \right] \\ &= (\varrho + \delta + \theta g) - f'(k_{et}) \\ \frac{\dot{c}_{et}}{c_{et}} &= \frac{1}{\theta} \left[ f'(k_{et}) - (\varrho + \delta + \theta g) \right] \end{aligned} \tag{31}$$

- Equation (31) is the continuous version of Euler equation.

# Continuous Time Growth Model

- The corresponding transversality condition is

$$
\begin{aligned}
0 &= \lim_{t\to\infty} \lambda_t k_{et} e^{-\varrho^* t} \\
0 &= \lim_{t\to\infty} (c_{et})^{-\theta} k_{et} e^{-[\varrho-n-(1-\theta)g]t}
\end{aligned}
$$

- Together with the transition equation of $k_t$, the following two differential equations and two boundary conditions solve the optimal growth path:

$$
\begin{aligned}
\frac{\dot{c}_{et}}{c_{et}} &= \frac{1}{\theta}\left[f'\left(k_{et}\right) - \left(\varrho + \delta + \theta g\right)\right], && (32) \\
\dot{k}_{et} &= f\left(k_{et}\right) - c_{et} - \left(g + n + \delta\right) k_{et}, k_{e0} \text{ is given.} && (33) \\
0 &= \lim_{t \to \infty} \left(c_{et}\right)^{-\theta} k_{et} e^{-[\varrho - n - (1-\theta) g] t}, && (34)
\end{aligned}
$$

- **The Phase Diagram:** One of the merits of working with a continuous model is that we can use the phase diagram. Let me describe what the phase diagram is. The first, let me define the steady state.

**Definition**

On the steady state the path of $(c_e^*, k_e^*)$ satisfies

$$\dot{c}_e^* = \dot{k}_e^* = 0.$$

- Therefore, the following equation must be satisfied on the steady sate:

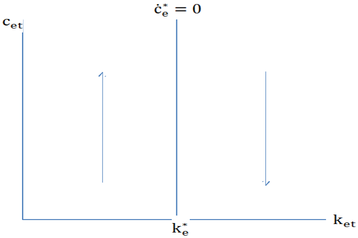
$$\dot{c}_e^* = 0 : f'(k_e^*) = \varrho + \delta + \theta g \tag{35}$$

$$\dot{k}_e^* = 0 : c_e^* = f(k_e^*) - (g + n + \delta) k_e^* \tag{36}$$

# Phase Diagram

- $\dot{c}_e^* = 0$: Note that $k_e^*$ is uniquely determined by equation (35).
  1. When $k_{et} = k_e^*$, $\dot{c}_e^* = 0$.
  2. When $k_{et} < k_e^*$, $f'(k_{et}) > \varrho + \delta + \theta g$. $\Rightarrow \dot{c}_{et} > 0$.
  3. When $k_{et} > k_e^*$, $f'(k_{et}) < \varrho + \delta + \theta g$. $\Rightarrow \dot{c}_{et} < 0$.

## Phase Diagram 1



# Phase Diagram

- Equation (36) implies that

$$\frac{dc_e^*}{dk_e^*} = f'\left(k_e^*\right) - \left(g + n + \delta\right), \quad \frac{d^2 c_e^*}{d\left(k_e^*\right)^2} = f''\left(k_e^*\right) < 0$$

- Hence, $c_e^*$ has the maximum value at

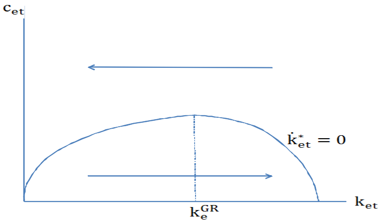
$$f'\left(k_e^{GR}\right) = g + n + \delta. \tag{37}$$

  - Capital stock, $k_e^{GR}$ is called the golden rule level of the capital stock. If you want to maximize $c_e$ when $\dot{k}_e^* = 0$ , equation (37) must be satisfied.

- $\dot{k}_e^* = \mathbf{0}$:

  1. When $c_e^* = f\left(k_{et}^*\right) - \left(g + n + \delta\right) k_{et}^*$, $\dot{k}_{et} = 0$
  2. When $c_{et} > f\left(k_{et}^*\right) - \left(g + n + \delta\right) k_{et}^*$, $\dot{k}_{et} < 0$.
  3. When $c_{et} < f\left(k_{et}^*\right) - \left(g + n + \delta\right) k_{et}^*$, $\dot{k}_{et} > 0$.

## Phase Diagram 2



# Phase Diagram

- Compare

$$f'(k_e^*) = \varrho + \delta + \theta g, \text{ and } f'\left(k_e^{GR}\right) = g + n + \delta.$$

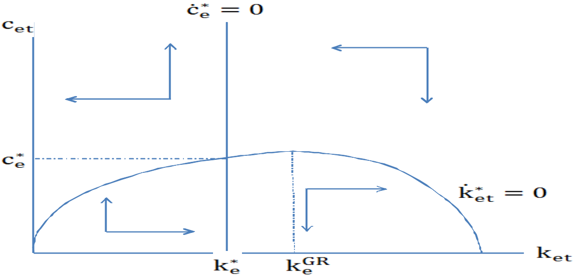
  - Note that

$$g + n + \delta < \varrho + \delta + \theta g \text{ iff } (\varrho - n) > (1 - \theta) g.$$

- Because we have assumed $\rho - n > (1 - \theta) g$, $f'(k_e^*) > f'\left(k_e^{GR}\right)$, $k_e^* < k_e^{GR}$.
  - Since the agent discounts future, it is not optimal to reduce current consumption to reach the golden rule level of the capital stock. The steady state value of capital stock, $k_e^*$, is called the modified golden rule level of capital stock.
  - Note that given the assumption of $\rho - n > (1 - \theta) g$,

$$0 = \lim_{t \to \infty} (c_e^*)^{-\theta} k_e^* e^{-[\varrho - n - (1-\theta) g] t}.$$

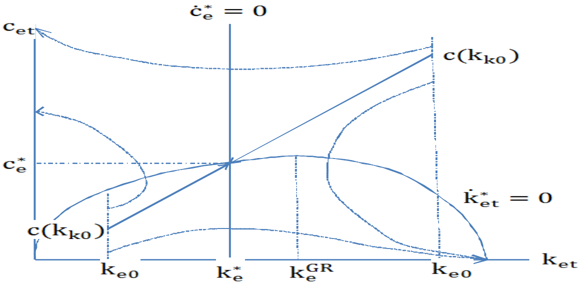
# Phase Diagram 3



# Phase Diagram

- **The saddle point path:** Note that given $k_{e0}$, there exists $c\left(k_{e0}\right)$ which converges to the steady state. If $c_{e0} = c\left(k_{e0}\right)$, economy will eventually reach the steady state. Since the steady state satisfies the transversality condition (34), this path is optimal. This path is called *the saddle point path*.

# Phase Diagram 4



# Phase Diagram

- If $c_{e0} > c(k_{e0})$, it is known that it hits $k_{et} = 0$ line in a finite time. But when $k_{et} = 0$, $c_{et}$ must jump to 0 from equation (33). Otherwise, $k_{et}$ becomes negative. However, the Euler equation (32) does not allow the jump. Hence this path violates the Euler equation (32).
- If $c_{e0} < c(k_{e0})$, the phase diagram says that this path eventually converges to the point $c_{et} = 0$ and $k_{et} < \infty$. We show that this path violates the transversality condition (34).

# Phase Diagram

- **Proof:** Because for large $t$, $k_{et} > k_{et}^{GR}$, $f'(k_{et}) < f'\left(k_e^{GR}\right)$. Hence

$$\begin{aligned}
\frac{\dot{c}_{et}}{c_{et}} &= \frac{1}{\theta}\left[f'(k_{et}) - (\varrho + \delta + \theta g)\right] \\
&< \frac{1}{\theta}\left[f'\left(k_e^{GR}\right) - (\varrho + \delta + \theta g)\right] \\
&= \frac{1}{\theta}\left[g + n + \delta - (\varrho + \delta + \theta g)\right] \\
&= \frac{1}{\theta}\left[(1-\theta)g + n - \varrho\right] = -\frac{\varrho^*}{\theta}
\end{aligned}$$

# Phase Diagram

- **Proof:** Hence, for large $t$

$$c_{et} < c_{e0} e^{-\frac{\varrho^*}{\theta} t}.$$

This implies that

$$(c_{et})^{-\theta} k_{et} e^{-\varrho^* t} > \left( c_{e0} e^{-\frac{\varrho^*}{\theta} t} \right)^{-\theta} k_{et} e^{-\varrho^* t} = c_{e0} k_{et} > 0$$

Hence,

$$0 < \lim_{t \to \infty} (c_{et})^{-\theta} k_{et} e^{-\varrho^* t}.$$

It proves that this path violates the transversality condition.

- In sum, the unique globally stable optimal path is characterized by the saddle point path. On this path, for any given $k_{e0}$, the agent optimally chooses $c\left(k_{e0}\right)$ with his expectation that economy will converge to the steady state.
- Although I only explain the basic idea about how to use Phase Diagram in this lecture, you can apply this method to the varieties of economic issues.

# Phase Diagram

- **Assignment:** Consider the following social planner's problem given government expenditure $G$:

$$\max_{c_t} \int_0^\infty e^{-\varrho t} U(c_t)\, dt$$
$$\dot{k}_t = f(k_t) - \delta k_t - c_t - g,$$

where $c_t$ is consumption, $g$ is constant government expenditure, $k_t$ is capital stock, $\varrho$ is the discount rate, $\delta$ is the depreciation rate of capital stock. Assume that the aggregate production function, $f(\cdot)$, satisfies usual assumptions: $f'(\cdot) > 0$, $f''(\cdot) < 0$, $f(0) = 0$, $f'(0) = \infty$, $f'(\infty) = 0$.

1. Draw and explain the Phase Diagram which summarizes the dynamics of $k_t$ and $c_t$
2. Discuss how government expenditure $g$ influences the steady state value of $k_t$ and $c_t$.

# Stochastic Dynamic Optimization

- So far, we don't have any uncertainty in the model.
- Introducing uncertainty in our model greatly increases the applicability of our method to several other issues such as a financial market, business cycle and labor search.
- In this section, we simply add uncertainty to the basic dynamic programming.

# Stochastic Dynamic Optimization

- We introduce random factors into the optimization problem.

$$\begin{aligned} U(S_\tau, \tau) &= \max_{\{X_t\}_\tau^\infty} E_\tau \left\{ \sum_{t=\tau}^{\infty} \beta^{(t-\tau)} r(X_t, S_t) \right\}, \\ s.t.\ S_{t+1} &= G(X_t, S_t, \varepsilon_{t+1}), S_\tau \text{ is given}, \end{aligned}$$

where $\varepsilon_{t+1}$ has a distribution function $Q(\varepsilon_{t+1})$.

- Note that

$$\begin{aligned}
U(S_\tau,\tau) &= \max_{\{X_t\}_\tau^\infty} E_\tau \left\{ \sum_{t=\tau}^{\infty} \beta^{(t-\tau)} r(X_t,S_t) \right\} \\
&= \max_{X_\tau} E_\tau \left\{ \begin{array}{c} r(X_\tau,S_\tau)+ \\ \beta \max_{\{X_t\}_{\tau+1}^\infty} E_{\tau+1} \sum_{t=\tau+1}^{\infty} \beta^{(t-(\tau+1))} r(X_t,S_t) \end{array} \right. \\
&= \max_{X_\tau} \{ r(X_\tau,S_\tau) + \beta E_\tau U(S_{\tau+1},\tau+1) \}
\end{aligned}$$

# Stochastic Dynamic Optimization

- The Bellman equation can be written as

$$V(S_t) = \max_{X_\tau} \left\{ r(X_t, S_t) + \beta \int V(G(X_t, S_t, \varepsilon_{t+1}))\, dQ(\varepsilon_{t+1}) \right\}$$

where

$$\begin{aligned} & \int V(G(X_t, S_t, \varepsilon_{t+1}))\, dQ(\varepsilon_{t+1}) \\ = & \int V(G(X_t, S_t, \varepsilon_{t+1}))\, Q'(\varepsilon_{t+1})\, d\varepsilon_{t+1} \quad or \\ = & \sum_{i=1}^{n} V(G(X_t, S_t, \varepsilon_i)) \Pr(\varepsilon_{t+1} = \varepsilon_i) \end{aligned}$$

and

$$\Pr(\varepsilon_{t+1} = \varepsilon_i) = Q(\varepsilon_i) - Q(\varepsilon_{i-1})$$

- We can analyze stochastic dynamic optimization problem using the method similar to the deterministic case.

- **Example:** Consider the following Lucas (1978) model of asset prices:
  1. Consider an economy consisting of a large number of identical agents.
  2. Each agent is born with one "tree". Each period, each tree yields fruits (=dividends) in the amount $d_t$ into its owner at the beginning of period $t$.
  3. Let $p(d_t)$ is the price of a tree in period $t$, which is measured in units of consumption goods per tree.

- **Household**

$$\max_{\{c_t\}_{t=\tau}^{\infty}} E_\tau \sum_{t=\tau}^{\infty} \beta^{(t-\tau)} U(c_t),$$

$$\begin{aligned} s.t.\ p(d_t)\, s_{t+1} + c_t &= [p(d_t) + d_t]\, s_t \\ d_{t+1} &= G(d_t, \varepsilon_{t+1}),\ \varepsilon_{t+1} \tilde{} Q(\varepsilon_{t+1}) \end{aligned}$$

where $s_t$ is the number of trees owned at the beginning of period, $t$.

- **Capital market clearing condition:** Because everything is identical, nobody sells and buys in an equilibrium.

$$s_t = 1$$

- Bellman Equation

$$
\begin{aligned}
V(s_t, d_t) &= \max_{c_t} \left\{ \begin{array}{c} U(c_t) + \\ \beta \int V(s_{t+1}, d_{t+1})\, dQ(\varepsilon_{t+1}) \end{array} \right\} \\
s.t.\ s_{t+1} &= \frac{[p(d_t) + d_t]\, s_t - c_t}{p(d_t)} \\
d_{t+1} &= G(d_t, \varepsilon_{t+1}),\ \varepsilon_{t+1} \tilde{}\, Q(\varepsilon_{t+1})
\end{aligned}
$$

- First Order Conditions:

$$U'(c_t) = \beta \int \frac{V_1(s_{t+1}, d_{t+1})}{p(d_t)} dQ(\varepsilon_{t+1})$$

- Envelope Theorem:

$$V_1(s_t, d_t) = \beta \int [p(d_t) + d_t] \frac{V_1(s_{t+1}, d_{t+1})}{p(d_t)} dQ(\varepsilon_{t+1})$$

- Euler Equation:
  1. Combining the first order condition and the envelope theorem
$$[p(d_t) + d_t]\, U'(c_t) = V_1(s_t, d_t)$$
  2. Using envelope theorem, Euler equation can be derived as
$$\begin{aligned} & [p(d_t) + d_t]\, U'(c_t) \\ = \; & \beta \int \frac{[p(d_t) + d_t]}{p(d_t)} [p(d_{t+1}) + d_{t+1}]\, U'(c_{t+1})\, dQ(\varepsilon_{t+1}) \end{aligned}$$
     Hence
$$U'(c_t) = \beta \int \frac{[p(d_{t+1}) + d_{t+1}]}{p(d_t)} U'(c_{t+1})\, dQ(\varepsilon_{t+1})$$

# Stochastic Dynamic Optimization

- TVC:

$$
\begin{aligned}
0 &= \lim_{t\to\infty} \beta^{(t-\tau)} E\left[V_1\left(s_t, d_t\right) s_t | d_\tau\right] \\
&= \lim_{t\to\infty} \beta^{(t-\tau)} E_\tau \left[\left[p\left(d_t\right) + d_t\right] U'\left(c_t\right) s_t\right]
\end{aligned}
$$

  - The operator $E_t\left[\cdot\right]$ is the conditional expectation given the information available at date $t$. Hence, stochastic version of the transversality condition demands an expectation conditioning on the information available at date $\tau$.

# Stochastic Dynamic Optimization

- Equilibrium Condition implies

$$s_t = 1 \Rightarrow c_t = d_t$$

- Euler equation and TVC are

$$\begin{aligned} U'(d_t) &= \beta \int \frac{[p(d_{t+1}) + d_{t+1}]}{p(d_t)} U'(d_{t+1})\, dQ(\varepsilon_{t+1}) \\ 0 &= \lim_{t \to \infty} \beta^{(t-\tau)} E_\tau \left[ [p(d_t) + d_t]\, U'(d_t) \right] \end{aligned}$$

# Stochastic Dynamic Optimization

- Euler Equation can be rewritten as

$$
\begin{aligned}
p(d_t) &= \int M_{t+1}\left[p(d_{t+1}) + d_{t+1}\right] dQ(\varepsilon_{t+1}) \\
&= E_t\left[M_{t+1}\left[p(d_{t+1}) + d_{t+1}\right]\right] \\
&= E\left[M_{t+1}\left[p(d_{t+1}) + d_{t+1}\right] | d_t\right]
\end{aligned}
$$

where $M_{t+1} = \beta \frac{U'(c_{t+1})}{U'(c_t)} = \beta \frac{U'(d_{t+1})}{U'(d_t)}$ is the *intertemporal marginal rate of substitution* also known as the *stochastic discount factor* or *pricing kernel*. Because information needed to predict future sequences of $d_{t+i}$ is summarized by $d_t$, the last equality must be satisfied.

# Stochastic Dynamic Optimization

- Note that

$$
\begin{aligned}
& p\left(d_t\right) \\
= \; & E_t\left[M_{t+1}\left[p\left(d_{t+1}\right)+d_{t+1}\right]\right] \\
= \; & E_t\left[M_{t+1} d_{t+1}\right]+E_t\left[M_{t+1} p\left(d_{t+1}\right)\right] \\
= \; & E_t\left[M_{t+1} d_{t+1}\right]+E_t\left[M_{t+1} E_{t+1}\left[M_{t+2}\left[p\left(d_{t+2}\right)+d_{t+2}\right]\right]\right] \\
= \; & E_t\left[M_{t+1} d_{t+1}\right]+E_t\left[M_{t+1} M_{t+2} d_{t+2}\right]+E_t\left[M_{t+1} M_{t+2} p\left(d_{t+2}\right)\right] \\
= \; & E_t\left[M_{t+1} d_{t+1}\right]+E_t\left[M_{t+1} M_{t+2} d_{t+2}\right] \\
& +E_t\left[M_{t+1} M_{t+2} E_{t+2}\left[M_{t+3}\left[p\left(d_{t+3}\right)+d_{t+3}\right]\right]\right] \\
= \; & E_t\left[M_{t+1} d_{t+1}\right]+E_t\left[M_{t+1} M_{t+2} d_{t+2}\right] \\
& +E_t\left[M_{t+1} M_{t+2} M_{t+3}\left[p\left(d_{t+3}\right)+d_{t+3}\right]\right] \\
= \; & \sum_{i=1}^{\infty} E_t\left[\Pi_{x=1}^i M_{t+x} d_{t+i}\right]+\lim_{i \rightarrow \infty} E_t\left[\Pi_{x=1}^i M_{t+x}\left[p\left(d_{t+i}\right)+d_{t+i}\right]\right]
\end{aligned}
$$

# Stochastic Dynamic Optimization

- Note that

$$\begin{aligned}\Pi_{x=1}^{i} M_{t+x} &= \beta \frac{U'(c_{t+1})}{U'(c_t)} \beta \frac{U'(c_{t+2})}{U'(c_{t+1})} \beta \frac{U'(c_{t+3})}{U'(c_{t+2})} .. \beta \frac{U'(c_{t+i})}{U'(c_{t+i-1})} \\ &= \beta^i \frac{U'(c_{t+i})}{U'(c_t)}\end{aligned}$$

Hence

$$\begin{aligned}&\lim_{i\to\infty} E\left[\Pi_{x=1}^{i} M_{t+x}\left[p(d_{t+i}) + d_{t+i}\right] | d_t\right] \\ = &\lim_{i\to\infty} E\left[\beta^i \frac{U'(c_{t+i})}{U'(c_t)}\left[p(d_{t+i}) + d_{t+i}\right] | d_t\right] \\ = &\frac{\lim_{i\to\infty} \beta^i E\left[U'(c_{t+i})\left[p(d_{t+i}) + d_{t+i}\right] | d_t\right]}{U'(c_t)} \\ = &\ 0\end{aligned}$$

# Stochastic Dynamic Optimization

- The share price is an expected discounted stream of dividends with a stochastic discount factor. .

$$p\left(d_t\right) = \sum_{i=1}^{\infty} E\left[\Pi_{x=1}^{i} M_{t+x} d_{t+i} \middle| d_t\right]$$

where

$$\Pi_{x=1}^{i} M_{t+x} = \beta^i \frac{U'\left(d_{t+i}\right)}{U'\left(d_t\right)}$$

- Euler Equation can be also rewritten as

$$1 = \int M_t \left(1 + \rho_t^r\right) dQ\left(\varepsilon_t\right) = E\left[M_t\left(1 + \rho_t^r\right)\right]$$

where $\rho_t^r = \frac{d_t + p(d_t) - p(d_{t-1})}{p(d_{t-1})}$ is the return from this asset.

# Stochastic Dynamic Optimization

## Lemma

$$Cov(X_t, Y_t) = E[X_t Y_t] - E[X_t] E[Y_t]$$

## Proof.

$$\begin{aligned}
& Cov(X_t, Y_t) \\
= \; & E[(X_t - E[X_t])(Y_t - E[Y_t])] \\
= \; & E[X_t Y_t - E[X_t] Y_t - X_t E[Y_t] + E[X_t] E[Y_t]] \\
= \; & E[X_t Y_t] - E[X_t] E[Y_t] - E[X_t] E[Y_t] + E[X_t] E[Y_t] \\
= \; & E[X_t Y_t] - E[X_t] E[Y_t]
\end{aligned}$$

□

- Using lemma,

$$\begin{aligned} Cov(M_t, 1+\rho_t^r) &= E\left[M_t\left(1+\rho_t^r\right)\right] - E\left[M_t\right] E\left[1+\rho_t^r\right] \\ E\left[M_t\right] E\left[1+\rho_t^r\right] &= E\left[M_t\left(1+\rho_t^r\right)\right] - Cov(M_t, 1+\rho_t^r) \\ E\left[1+\rho_t^r\right] &= \frac{1}{E\left[M_t\right]}\left[E\left[M_t\left(1+\rho_t^r\right)\right] - Cov(M_t, 1+\rho_t^r)\right] \\ &= \frac{1}{E\left[M_t\right]}\left[1 - Cov(M_t, 1+\rho_t^r)\right] \end{aligned}$$

because of $E\left[M_t\left(1+\rho_t^r\right)\right] = 1$.

- Hence we can get

$$\begin{aligned} E\left[1+\rho_t^r\right] &= \frac{1}{E\left[M_t\right]}\left[1 - Cov(M_t, 1+\rho_t^r)\right] \\ E\left[\rho_t^r\right] &= \frac{1}{E\left[M_t\right]}\left[1 - Cov(M_t, 1+\rho_t^r)\right] - 1 \qquad (38) \end{aligned}$$

- **Risk free rate of return:** If there is a risk free asset, the return on this asset $\rho$ must satisfy

$$Cov(M_t, 1+\rho) = 0$$

Hence, equation (38) implies

$$1+\rho = \frac{1}{E[M_t]}$$

- **Return on the Risky Asset:** Substituting the risk free rate into equation (38),

$$\begin{aligned} E\left[1+\rho_t^r\right] &= (1+\rho)\left[1 - Cov(M_t, 1+\rho_t^r)\right] \\ &= (1+\rho) - (1+\rho)\, Cov(M_t, 1+\rho_t^r) \\ E\left[\rho_t^r\right] &= \rho - (1+\rho)\, Cov(M_t, 1+\rho_t^r) \end{aligned}$$

- A stock return must be equal to risk free rate of return +Risk premium. The risk premium is now a decreasing function of covariance between a return and $M_t$. Why? Note that because $M_t = \beta \frac{U'(c_t)}{U'(c_{t-1})}$,

$$c_t \uparrow = U'(c_t) \downarrow = M_t \downarrow$$

1. Suppose that covariance is negative. We expect a high return when $M_t$ is low, that is, the consumption is high. We expect a low return when the consumption is low. So you cannot smooth your consumption with this asset much. Therefore, the risk premium is high.
2. Suppose that covariance is positive. We expect a high return when $M_t$ is high, that is, the consumption is low. We expect a low return when the consumption is high. So you can smooth your consumption with this asset. Therefore, the risk premium is low.

- This model is called Consumption Capital Asset Pricing Model (Consumption CAPM).

# Stochastic Dynamic Optimization

- **Assignment:** Consider the following investment problem:

$$\max_{I_t, L_t} E_0 \sum_{t=0}^{\infty} \Pi_{x=0}^{t} M_x \left[ z_t F\left(K_t, L_t\right) - wL_t - C\left(I_t, K_t\right) - I_t \right]$$

$$K_t = I_t + (1-\delta) K_t$$

$$z_{t+1} = z_t + \varepsilon_{t+1}, \ \varepsilon_{t+1} \tilde{} Q\left(\varepsilon_{t+1}\right)$$

where $F\left(K_t, L_t\right)$ and $C\left(I_t, K_t\right)$ are constant returns to scale, and $M_t$ is the stochastic discount factor. (For this exercise, the sequence of $M_t$ is taken as given. ). Suppose that there exists $B$ such that $0 \leq M_t \left( \frac{I_t}{K_t} + (1-\delta) \right) \leq B < 1$ for any $\frac{I_t}{K_t}$.

1. Define the Bellman Equation of this problem.
2. Show that the value function is linear in $K_t$.
3. Derive Euler Equation of this problem.

# Rational Expectation and GMM

- Euler equation often gives us $E_t\left(X_{t+s}\left(\psi_x\right)\right) = Y_t\left(\psi_y\right)$, where $X_{t+s}\left(\psi_x\right)$ and $Y_t\left(\psi_y\right)$ are a $(K \times 1)$ column vector which depends on $\psi = \left(\psi_x, \psi_y\right)$.
  - Example, suppose that $U(c) = \frac{(c)^{(1-\theta)}-1}{1-\theta}$. Then, $U'(c) = c^{-\theta}$. Hence, Euler equation of consumption CAPM can be summarized by

$$1 = E_t\left[M_{t+1}\left(1+\rho^r_{t+1}\right)\right]$$

where $M_{t+1} = \beta\frac{U'(c_{t+1})}{U'(c_t)} = \beta\frac{(c_{t+1})^{-\theta}}{(c_t)^{-\theta}} = \beta\left(\frac{c_t}{c_{t+1}}\right)^{\theta}$. In this case, $K = 1$

$$\begin{aligned} X_{t+s}\left(\psi_x\right) &= \beta\left(\frac{c_t}{c_{t+1}}\right)^{\theta}\left(1+\rho^r_{t+1}\right) \\ Y_t\left(\psi_y\right) &= 1,\ \psi_x = (\beta, \theta) \end{aligned}$$

- Define $u_{t+s}(\psi) = Y_t\left(\psi_y\right) - X_{t+s}(\psi_x)$. Then

$$
\begin{aligned}
E_t\left[u_{t+s}(\psi)\right] &= E_t\left[Y_t\left(\psi_y\right) - X_{t+s}(\psi_x)\right] \\
&= Y_t\left(\psi_y\right) - E_t\left[X_{t+s}(\psi_x)\right] \\
&= 0.
\end{aligned}
$$

We can interpret $u_{t+s}(\psi)$ as a prediction error. Take any known variable at date $t$, $z_j$, where $j = 1, .., J$. Then

$$E\left(z_j u_{t+s}(\psi)\right) = E\left[z_j E_t\left[u_{t+s}(\psi)\right]\right] = 0 \ \forall j$$

  - This is called an orthogonality condition.

# Rational Expectation and GMM

- Given the orthogonality condition, for large $N$

$$\lim_{N\to\infty} \frac{\sum_i^N z_j^i u_{t+s}^i(\psi)}{N} = E\left[z_j u_{t+s}(\psi)\right] = 0, \ \forall j$$

- If the number of orthogonality conditions is equal to the number of $\psi$, GMM estimate $\psi^*$ by

$$0 = \frac{\sum_i^N z_j^i u_{t+s}^i(\psi^*)}{N}$$

  - This is equivalent to a non-linear IV estimation: if there is 1 orthogonality condition and 1 parameter $\psi_x$ with $u_{t+s}^i(\psi) = y_t^i - \psi_x x_{t+s}^i$,

$$0 = \frac{\sum_i^N z_j^i\left(y_t^i - \psi_x^* x_{t+s}^i\right)}{N}$$

$$\psi_x^* \frac{\sum_i^N z^i x_{t+s}^i}{N} = \frac{\sum_i^N z^i y_t^i}{N} \Rightarrow \psi_x^* = \left[\sum_i^N z^i x_{t+s}^i\right]^{-1} \sum_i^N z^i y_t^i$$

# Rational Expectation and GMM

- In general, the number of orthogonality conditions is not equal to the number of parameters, $\psi$.
  1. If the number of orthogonality condition is smaller than the number of parameters, we cannot identify the parameters.
  2. If the number of orthogonality condition is greater than the number of parameters, we have the overidentification of the parameters.
- Define a $(KJ \times 1)$ vector, $g(N:\psi) = \begin{pmatrix} \frac{\sum_i^N z_1^i u_{t+s}^i(\psi)}{N} \\ .. \\ \frac{\sum_i^N z_J^i u_{t+s}^i(\psi)}{N} \end{pmatrix}$. When the parameters are overidentified, GMM estimates $\psi^*$ by

$$\psi^* = \arg\min_{\psi} g(N:\psi)' \mathbf{W}_N^{-1} g(N:\psi)$$

where $\mathbf{W}_N^{-1}$ is a $(KJ \times KJ)$ optimal weighting matrix, which can be estimated from data.

# Stochastic Growth Model

- In this section, we introduce a stochastic shock into the neoclassical growth model.
- In addition to the shock, we allow that household can value leisure.
- These framework plays the benchmark for studying business cycle.

- The general problem is as follows:

$$\begin{aligned} &\max_{\{c_t, l_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t N_t U(c_t, 1 - l_t) \\ s.t.\ K_{t+1} &= z_t F(K_t, T_t l_t N_t) + (1 - \delta) K_t - c_t N_t,\ \ K_0 \ is\ given \\ T_{t+1} &= (1 + g)\, T_t,\ \ T_0 \ is\ given \\ N_{t+1} &= (1 + n)\, N_t,\ \ N_0 \ is\ given \\ \ln z_{t+1} &= \zeta \ln z_t + \varepsilon_{t+1},\ \ z_0 \ is\ given,\ \ \varepsilon_{t+1} \tilde{}\, Q\,(\varepsilon_{t+1}) \end{aligned}$$

# Stochastic Growth Model

- **Assignment:** Assume that the utility exhibits CRRA:

$$U\left(c_{et}T_t, l_t\right) = \frac{\left[c_{et}T_t v\left(1-l_t\right)\right]^{1-\theta} - 1}{1-\theta}, \theta \geq 0$$

  Show that the original problem can be rewritten as

$$\max_{\{c_{et}, l_t\}} E_0 \sum_{t=0}^{\infty} \left(\beta^*\right)^t \tilde{u}\left(c_{et}, 1-l_t\right)$$

$$\begin{aligned} s.t.\ k_{et+1} &= \frac{z_t F\left(k_{et}, l_t\right) + \left(1-\delta\right) k_{et} - c_{et}}{\left(1+g\right)\left(1+n\right)}.\ k_{e0}\ is\ given \\ \ln z_{t+1} &= \zeta \ln z_t + \varepsilon_{t+1},\ z_0\ is\ given,\ \varepsilon_{t+1} \tilde{}\, Q\left(\varepsilon_{t+1}\right) \end{aligned}$$

  where $\tilde{u}\left(c_{et}, 1-l_t\right) = \frac{\left[c_{et} v\left(l_t\right)\right]^{1-\theta}}{1-\theta}$ if $0 \leq \theta \neq 1$, $\tilde{u}\left(c_{et}, 1-l_t\right) = \ln c_{et} + \ln v\left(1-l_t\right)$ if $\theta = 1$ and $\beta^* = \beta\left(1+n\right)\left(1+g\right)^{(1-\theta)}$.

- Assuming that $\beta^* < 1$, the following Bellman Equation can be defined.

$$
\begin{aligned}
V(k_{et}, z_t) &= \max_{\{c_{et}, l_t\}} \left\{ \tilde{u}(c_{et}, 1 - l_t) + \beta^* \int V(k_{et+1}, z_{t+1})\, dQ(\varepsilon_{t+1}) \right\} \\
s.t.\ k_{et+1} &= \frac{z_t F(k_{et}, l_t) + (1-\delta) k_{et} - c_{et}}{(1+g)(1+n)} \\
\ln z_{t+1} &= \zeta \ln z_t + \varepsilon_{t+1},
\end{aligned}
$$

# Stochastic Growth Model

- First Order Conditions

$$\tilde{u}_1(c(k_{et}, z_t), 1 - l(k_{et}, z_t)) = \frac{\beta^* \int V_1(\kappa(k_{et}, z_t), z_{t+1})\, dQ(\varepsilon_{t+1})}{(1+g)(1+n)}$$

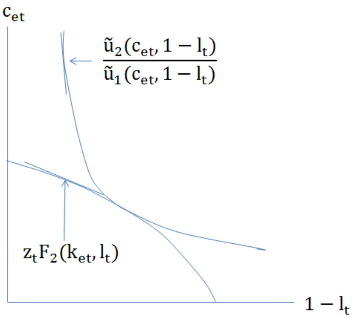
$$\tilde{u}_2(c(k_{et}, z_t), 1 - l(k_{et}, z_t)) = \frac{\beta^* \int V_1(\kappa(k_{et}, z_t), z_{t+1})\, dQ(\varepsilon_{t+1})\, z_t F_2(k_{et}, l(k_{et}, z_t))}{(1+g)(1+n)}$$

where $\kappa(k_{et}, z_t) = \frac{z_t F(k_{et}, l(k_{et}, z_t)) + (1-\delta)k_{et} - c(k_{et}, z_t)}{(1+g)(1+n)}$.

- Hence, consumption and leisure relationship is influenced by

$$\frac{\tilde{u}_2(c_{et}, 1 - l_t)}{\tilde{u}_1(c_{et}, 1 - l_t)} = z_t F_2(k_{et}, l_t)$$

# Intratemporal Choice Between Consumption and Leisure



- Envelope theorem

$$V_1\left(k_{et}, z_t\right) = \frac{\begin{array}{c}\beta^* \int V_1\left(\kappa\left(k_{et}, z_t\right), z_{t+1}\right) dQ\left(\varepsilon_{t+1}\right) \\ \left[z_t F_1\left(k_{et}, l\left(k_{et}, z_t\right)\right) + 1 - \delta\right]\end{array}}{(1+g)(1+n)}$$

- Euler Equation
  1. Combining the first order condition and envelope theorem,

$$V_1\left(k_{et}, z_t\right) = \tilde{u}_1(c_{et}, 1 - l_t)\left[z_t F_1\left(k_{et}, l_t\right) + 1 - \delta\right]$$

  2. Using envelope theorem

$$= \frac{\begin{array}{c}\tilde{u}_1(c_{et}, 1 - l_t)\left[z_t F_1\left(k_{et}, l_t\right) + 1 - \delta\right] \\ \beta^* \int \tilde{u}_1(c_{et+1}, 1 - l_{t+1})\left[z_{t+1} F_1\left(k_{et+1}, l_{t+1}\right) + 1 - \delta\right] dQ\left(\varepsilon_{t+1}\right) \\ \left[z_t F_1\left(k_{et}, l_t\right) + 1 - \delta\right]\end{array}}{(1 + g)(1 + n)}$$

Hence,

$$\begin{aligned} & \tilde{u}_1(c_{et}, 1 - l_t) \\ = & \int \beta^* \tilde{u}_1(c_{et+1}, 1 - l_{t+1}) \frac{\left[z_{t+1} F_1\left(k_{et+1}, l_{t+1}\right) + 1 - \delta\right]}{(1 + g)(1 + n)} dQ\left(\varepsilon_{t+1}\right) \\ = & \int \beta \tilde{u}_1(c_{et+1}, 1 - l_{t+1}) \frac{\left[z_{t+1} F_1\left(k_{et+1}, l_{t+1}\right) + 1 - \delta\right]}{(1 + g)^{\theta}} dQ\left(\varepsilon_{t+1}\right) \end{aligned}$$

# Stochastic Growth Model

- Euler Equation

$$
\begin{aligned}
& \tilde{u}_1(c_{et}, 1 - l_t) \\
= & \int \beta \tilde{u}_1(c_{et+1}, 1 - l_{t+1}) \frac{[z_{t+1} F_1(k_{et+1}, l_{t+1}) + 1 - \delta]}{(1+g)^\theta} dQ(\varepsilon_{t+1})
\end{aligned}
$$

- TVC

$$
\begin{aligned}
0 &= \lim_{t \to \infty} (\beta^*)^t E_0 [V_1(k_{et}, z_t) k_{et}] \\
&= \lim_{t \to \infty} (\beta^*)^t E_0 [\tilde{u}_1(c_{et}, 1 - l_t) [z_t F_1(k_{et}, l_t) + 1 - \delta] k_{et}]
\end{aligned}
$$

- **Summary Dynamics** $(c_{et}, l_t, k_{et}, z_t)$
  - Intratemporal substitution between leisure and consumption

$$\frac{\tilde{u}_2(c_{et}, 1 - l_t)}{\tilde{u}_1(c_{et}, 1 - l_t)} = z_t F_2(k_{et}, l_t)$$

  - Euler Equation and TVC

$$\begin{aligned} & \tilde{u}_1(c_{et}, 1 - l_t) \\ = & \int \beta \tilde{u}_1(c_{et+1}, 1 - l_{t+1}) \frac{[z_{t+1} F_1(k_{et+1}, l_{t+1}) + 1 - \delta]}{(1+g)^{\theta}} dQ(\varepsilon_{t+1}) \\ 0 = & \lim_{t \to \infty} (\beta^*)^t E_0 [\tilde{u}_1(c_{et}, l_t) [z_t F_1(k_{et}, 1 - l_t) + 1 - \delta] k_{et}] \end{aligned}$$

where $\beta^* = \beta(1+n)(1+g)^{(1-\theta)}$.

- **Summary Dynamics** $(c_{et}, l_t, k_{et}, z_t)$
  - Capital accumulation and the dynamics of productivity shocks.

$$k_{et+1} = \frac{z_t F(k_{et}, l_t) + (1-\delta) k_{et} - c_{et}}{(1+g)(1+n)}. \ k_{e0} \ \textit{is given}$$

$$\ln z_{t+1} = \zeta \ln z_t + \varepsilon_{t+1}, \ z_0 \ \textit{is given}, \ \varepsilon_{t+1} \tilde{} \, Q(\varepsilon_{t+1})$$

# Numerical Analysis

- With their seminal analysis of business cycle, Kidland and Prescott (1982) develops a new method of empirical work, calibration, in macroeconomics. In order to understand its spirits, let Kidland and Prescott (1996) speak how computational experiment can be implemented, in particular with a stochastic growth model.
- According to Kidland and Prescott (1996), an economic computational experiment has four steps.
  1. Pose a Question
  2. Use a well-tested theory
  3. Construct a model economy
  4. Calibrate the model economy
  5. Run the experiment

# Numerical Analysis

- **Pose a question:** "The purpose of a computational experiment is to derive a quantitative answer to some well-posed question. Thus, the first step in carrying out a computational experiment is to pose such a question." Kidland and Prescott (1996) Question may involve
  1. Assessments of the theoretical implications of changes in policy.
  2. the assessments of the ability of a model mimic features of the actual economy.
- In case of a basic real business cycle model, Kidland and Prescott (1996) poses
  - "Does a model designed to be consistent with long-term economic growth produce the sort of fluctuation that we associate with the business cycle?"
  - "How much would the U.S. postwar economy have fluctuated if technology shocks had been the only source of fluctuations?"

# Numerical Analysis

- **Use a well-tested theory:** "With a particular question in mind, a researcher needs some strong theory to carry out a computational experiment: that is, a researcher needs a theory that has been tested through use and found to provide reliable answers to a class of questions. Here, by theory we do not mean a set of assertions about the actual economy. Rather, following Lucas (1980), economic theory is defined to be "an explicit set of instructions for building... a mechanical imitation system" to answer a question" Kidland and Prescott (1996)
- "The basic theory used in the modern study of business cycles is the neoclassical growth model." Kidland and Prescott (1996)

- **Construct a Model Economy:** "With a particular theoretical framework in mind, the third step in conducting a computational experiment is to construct a model economy. Here, key issues are the amount of detail and the feasibility of computing the equilibrium process." Kidland and Prescott (1996)
  - "..an abstraction can be judged only relative to some given question. To criticize or reject a model because it is an abstraction is foolish: all models are necessarily abstractions. A model environment must be selected based on the question being addressed." Kidland and Prescott (1996)
  - "The selection and construction of a particular model economy should not depend on th answer provided. In fact, searching within some parametric class of economies for the one that best fits a set of aggregate time series makes little sense, because it isn't likely to answer an interesting question." Kidland and Prescott (1996)

- In case of a basic real business cycle model, they add a productivity shock and the choice of leisure to the neoclassical growth model because two-thirds of aggregate output are attributable to fluctuations of labor.

- **Calibrate the Model Economy:** "Now that a model has been constructed, the fourth step in carrying out a computational experiment is to calibrate that model. Originally, in physical sciences, calibration referred to the graduation of measuring instrument." Kidland and Prescott (1996)
    - "Generally, some economic questions have known answers, and the model should give an approximately correct answer to them if we are to have any confidence in the answer given to the question with unknown answer. Thus, data are used to calibrate the model economy so that it mimics the world as closely as possible along a limited, but clearly specified, number of dimensions. Kidland and Prescott (1996)

# Numerical Analysis

- **Calibrate the Model Economy:** In case of a standard business cycle model,
  1. the parameters must meet the long run averages of certain statistics such as the share of output paid to labor and the fraction of available hours worked per household, both of which have been remarkably stable over time.
  2. In addition, empirical results obtained in micro studies conducted at the individual or household level are often used as a means of pinning down certain parameters.

- **Calibrate the Model Economy:** The difference between estimation and calibration is also emphasized.
  - "Note that calibration is not an attempt at assessing the size of something: it is not estimation," "It is important to emphasize that the parameter values selected are not the ones that provide the best fit in some statistical sense. In some cases, the presence of a particular discrepancy between the data and the model economy is a test of the theory being used." Kidland and Prescott (1996)
  - Prescott (1986) also claims that "The models constructed within this theoretical framework are necessarily highly abstract. Consequently, they are necessarily false, and statistical hypothesis testing will reject them. This does not imply, however, that nothing can be learned from such quantitative theoretical exercises."

# Numerical Analysis

- **Run the Experiment:** "The fifth and final step in conducting a computational experiment is to run the experiment." Kidland and Prescott (1996)
  - "If the model economy has no aggregate uncertainty, then it is simply a matter of comparing the equilibrium path of the model economy with the path of the actual economy."
  - "If the model economy has aggregate uncertainty, first a set of statistics summarize relevant aspects of the behavior of the actual economy is selected. Then the computational experiment is used to generate many independent realizations of the equilibrium process for the model economy. In this way, the sampling distribution of this set of statistics can be determined to any degree of accuracy for the model economy with the value of the set of statistics for the actual economy." Kidland and Prescott (1996)
- A standard business cycle exercise typically compares the second moment such as the variance and the covariance of variables.

# Numerical Analysis

- Cooley and Prescott (1995) describes the process of calibration more precisely. They identify three steps.
  1. The first step is to restrict a model to a parametric class.
  2. to construct a set of measurements that are consistent with the parametric class of models.
  3. The third step is to assign values to the parameters of our models.
- Let us conduct these steps using our models.

- **Restricting the model:** we need to restrict technology and preference.
  - We have already restricting a part of our model to meet the balanced growth path: $\tilde{u}(c_{et}, 1 - l_t) = \frac{[c_{et} v(1-l_t)]^{1-\theta}}{1-\theta}$ if $\theta \neq 1$, $\tilde{u}(c_{et}, l_t) = \ln c_{et} + \ln v(1 - l_t)$ if $\theta = 1$. Cooley and Prescott (1995) assumes that $\theta = 1$ and $\ln v(1 - l) = \eta \ln(1 - l)$. Therefore, $\tilde{u}(c_{et}, 1 - l_t) = \ln c_{et} + \eta \ln(1 - l_t)$.
  - Knowing that labor share and capital share is stable, it is common to assume $F(k_e, l) = k_e^{\alpha} l^{1-\alpha}$. Because it is known that $\alpha = \frac{rK}{Y} = 1 - \frac{wL}{Y}$, even if the economy is on the transition path, the share must be stable.
  - We assume that $Q(\varepsilon_{t+1})$ is normally distributed with the mean 0 and the standard deviation of $\sigma_{\varepsilon}$.

# Numerical Analysis

- Note that $\tilde{u}\left(c_{et}, 1-l_t\right) = \ln c_{et} + \eta \ln\left(1-l_t\right)$ and $y_{et} = z_t F\left(k_e, l\right) = z_t k_e^{\alpha} l^{1-\alpha}$ implies $\tilde{u}_1\left(c_{et}, 1-l_t\right) = \frac{1}{c_{et}}$, $\tilde{u}_2\left(c_{et}, 1-l_t\right) = \frac{\eta}{1-l_t}$, $z_t F_1\left(k_{et}, l_t\right) = \alpha z_t k_{et}^{\alpha-1} l_t^{1-\alpha} = \alpha \frac{y_{et}}{k_{et}}$ and $z_t F_2\left(k_{et}, l_t\right) = \left(1-\alpha\right) z_t k_{et}^{\alpha} l_t^{-\alpha} = \left(1-\alpha\right) \frac{y_{et}}{l_t}$. Therefore, our economy is summarized by the following equations
- Intratemporal substitution between leisure and consumption

$$
\begin{aligned}
\frac{\tilde{u}_2\left(c_{et}, 1-l_t\right)}{\tilde{u}_1\left(c_{et}, 1-l_t\right)} &= z_t F_2\left(k_{et}, l_t\right) \\
\frac{\eta c_{et}}{1-l_t} &= \left(1-\alpha\right) \frac{y_{et}}{l_t}
\end{aligned}
$$

where $y_{et} = z_t k_{et}^{\alpha} l_t^{1-\alpha}$.

# Numerical Analysis

- Euler Equation and TVC

$$
\begin{aligned}
& \tilde{u}_1(c_{et}, 1 - l_t) \\
= & \int \beta \tilde{u}_1(c_{et+1}, 1 - l_{t+1}) \frac{\left[z_{t+1} F_1\left(k_{et+1}, l_{t+1}\right) + 1 - \delta\right]}{(1+g)^{\theta}} dQ\left(\varepsilon_{t+1}\right)
\end{aligned}
$$

$$
\frac{1}{c_{et}} = \int \frac{\beta}{c_{et+1}} \frac{\left[\alpha \frac{y_{et+1}}{k_{et+1}} + 1 - \delta\right]}{1+g} dQ\left(\varepsilon_{t+1}\right)
$$

$$
\begin{aligned}
0 &= \lim_{t \to \infty} \left(\beta^*\right)^t E_0 \left[\tilde{u}_1(c_{et}, l_t) \left[z_t F_1\left(k_{et}, 1 - l_t\right) + 1 - \delta\right] k_{et}\right] \\
&= \lim_{t \to \infty} \left[\beta (1+n)\right]^t E_0 \left[\frac{\left[\alpha \frac{y_{et}}{k_{et}} + 1 - \delta\right] k_{et}}{c_{et}}\right]
\end{aligned}
$$

where $y_{et} = z_t k_{et}^{\alpha} l_t^{1-\alpha}$.

# Numerical Analysis

- Capital accumulation and the dynamics of productivity shocks.

$$
\begin{aligned}
k_{et+1} &= \frac{y_{et} + (1-\delta) k_{et} - c_{et}}{(1+g)(1+n)}. \ k_{e0} \ \textit{is given} \\
\ln z_{t+1} &= \zeta \ln z_t + \varepsilon_{t+1}, \ z_0 \ \textit{is given}, \ \varepsilon_{t+1} \tilde{} N\left(0, \sigma_\varepsilon^2\right)
\end{aligned}
$$

where $y_{et} = z_t k_{et}^{\alpha} l_t^{1-\alpha}$.

# Numerical Analysis

- **Defining Consistent Measurements:** Cooley and Prescott (1995) was a quite careful about the measurement of capital stock, and output. Because the model is abstract, there is no treatment of government sector, household production, foreign sector and inventory. Therefore, the model economy's capital $K$ includes capital used in all these sectors plus the stock of inventory. Similarly, output includes the output productivity by all of this capital. They did some imputation. I discuss how they impute the values.
  1. Estimate the return on asset $\rho_t$ from private capital stock.
  2. They estimate depreciation rate $\delta_i$ separately for consumer durables and government sector.
  3. Using the estimated $\rho$ and $\delta_i$, they estimate the service flow for consumer durables and government sector, $r_i K_t^i = (\rho_t + \delta_i) K_t^i$

# Numerical Analysis

- They estimate the return on asset $\rho_t$ from private capital stock. Note that $\rho_t = \frac{rK_p - \delta K_p}{K_p}$ where $K_p$ is private capital stock.
  - $K^p$ is estimated by (1) the net stock of fixed reproducible private capital (not including the stock of consumer durables) + (2) the stock of inventories and (3) the stock of land where (1) is from Musgrave (1992), (2) is from the NIPA and (3) is from the Flow of Funds Accounts.
  - $rK^p$ is estimated by unambiguous capital income + $\alpha_p \times$ambiguous capital income $+\delta K$ where $\alpha_p$ is estimated to satisfy $\alpha_p = \frac{the\ \ estimated\ \ rK_p}{GDP}$, .$\delta K$ is estimated by GNP-NNP (consumption of fixed capital), unambiguous capital income is the sum of rental income, corporate profits and net interest, and ambiguous capital income is sum of proprietors income + net national product - national income.

# Numerical Analysis

- Next they estimate depreciation rate for government sector and consumer durable $\delta_i$ by the following equation

$$\frac{K^i_{t+1}}{Y_{t+1}}\frac{Y_{t+1}}{Y_t} = (1-\delta_i)\frac{K^i_t}{Y_t} + \frac{I^i_t}{Y_t}$$

  where $i$ is consumer durables or government sector and $Y_t$ is GNP. For consumer durables, $\left\{I^i_t\right\}$ is consumption of consumer durable as reported in the NIPA. For government investment $\left\{I^i_t\right\}$ is also from NIPA. The capital stock $\left\{K^i_t\right\}$ for consumer durable and government investment are taken from Musgrave (1992).

- Finally, they estimate the service flows of capital from the following equation.

$$r_i K^i_t = (\rho_t + \delta_i) K^i_t$$

# Numerical Analysis

- **Calibrating a Specific Model Economy:** Remember that our economy is summarized by 4 equations with suitable boundary conditions

$$
\begin{aligned}
\frac{\eta c_{et}}{1-l_t} &= (1-\alpha)\frac{y_{et}}{l_t} \\
\frac{1}{c_{et}} &= \int \frac{\beta}{c_{et+1}} \frac{\left[\alpha \frac{y_{et+1}}{k_{et+1}} + 1 - \delta\right]}{1+g} dQ\left(\varepsilon_{t+1}\right) \\
k_{et+1} &= \frac{y_{et} + (1-\delta) k_{et} - c_{et}}{(1+g)(1+n)}, \text{ given } k_{e0} \\
\ln z_{t+1} &= \zeta \ln z_t + \varepsilon_t, \text{ given } z_0 \ \varepsilon_{t+1} \tilde{} N\left(0, \sigma_\varepsilon^2\right)
\end{aligned}
$$

where $y_{et} = z_t k_{et}^{\alpha} l_t^{1-\alpha}$ and $0 = \lim_{t\to\infty} \left[\beta(1+n)\right]^t E_0 \left[\frac{\left[\alpha \frac{y_{et}}{k_{et}} + 1 - \delta\right] k_{et}}{c_{et}}\right]$.

Note that $y_{et}$, $c_{et}$ and $k_{et}$ are interpreted as the detrended GNP, Consumption and Capital per capita.

- We need to calibrate $\alpha$, $\beta$, $\delta$, $\eta$, $n$, $g$, $\zeta$ and $\sigma_\varepsilon$.

# Numerical Analysis

- $g = 0.0156$...measured by the time series average of rate of growth of real GNP per capita (On the balanced growth, $g_{\frac{Y}{N}} = g$).
- $n = 0.012$...average population growth.
- $\alpha = 0.4$...capital share is calibrated by the time series average of

$$\alpha = \frac{r_t K_t}{Y_t} = \frac{r_p K_t^p + r_c K_t^c + r_g K_t^g}{GNP + r_c K_t^c + r_g K_t^g}$$

# Numerical Analysis

- $\zeta = 0.95$ and $\sigma_\varepsilon = 0.0007$: Note that

$$\ln z_t = \ln Y_t - \alpha \ln K_t - (1-\alpha) \ln l_t T_t N_t$$

It means that

$$\begin{aligned} & \ln z_{t+1} - \ln z_t \\ = & \left(\ln Y_{t+1} - \ln Y_t\right) - \alpha \left(\ln K_{t+1} - \ln K_t\right) \\ & - (1-\alpha)\left(\ln l_{t+1} N_{t+1} - \ln l_t N_t\right) - (1-\alpha)\left(\ln T_{t+1} - \ln T_t\right) \\ = & \left(\ln Y_{t+1} - \ln Y_t\right) - \alpha \left(\ln K_{t+1} - \ln K_t\right) \\ & - (1-\alpha)\left(\ln l_{t+1} N_{t+1} - \ln l_t N_t\right) - (1-\alpha)\, g \end{aligned}$$

So given $\alpha$ and $g$ we can generate a series of $\ln z_{t+1} - \ln z_t$ and therefore $\ln z_t$ given $\ln z_0$. (Cooley and Prescott (1995) actually report that they measure it under the assumption of $g = 0$, though.) These sequences are used to calibrate $\zeta$ and $\sigma_\varepsilon$ of

$$\ln z_{t+1} = \zeta \ln z_t + \varepsilon_{t+1},\ \ \varepsilon_{t+1} \tilde{}\, N\left(0, \sigma_\varepsilon^2\right)$$

- In order to calibrate the remaining parameters $\beta$, $\delta$, and $\eta$. We use more theory and stylized facts on the balanced growth.
- $\delta = 0.012$ : On the steady state, capital accumulation equation implies

$$
\begin{aligned}
k_e^* &= \frac{y_e^* + (1-\delta) k_e^* - c_e^*}{(1+g)(1+n)} \\
(1+g)(1+n) k_e^* &= (1-\delta) k_e^* + i_e^* \\
(1-\delta) k_e^* &= (1+g)(1+n) k_e^* - i_e^* \\
1-\delta &= (1+g)(1+n) - \frac{i_e^*}{k_e^*} \\
\delta &= \frac{I}{K} + 1 - (1+g)(1+n)
\end{aligned}
$$

Given the measured values of $g$ and $n$, Cooley and Prescott (1995) calibrate $\delta$ using this equation.

# Numerical Analysis

- $\beta = 0.987$: If there is no uncertainty, Euler equation becomes

$$\frac{1}{c_{et}} = \frac{\beta}{c_{et+1}} \frac{\left[\alpha \frac{y_{et+1}}{k_{et+1}} + 1 - \delta\right]}{1+g}$$

On the steady state,

$$\begin{aligned}\frac{1}{\beta} &= \frac{\alpha \frac{y_e^*}{k_e^*} + 1 - \delta}{1+g} \\ &= \frac{\alpha \frac{Y}{K} + 1 - \delta}{1+g}\end{aligned}$$

Given the measured value of $\delta$, Cooley and Prescott (1995) calibrate $\beta$ using this equation.

- $\eta = 1.78$: Intratemporal substitution between consumption and leisure implies

$$\begin{aligned} \frac{\eta c_{et}}{1 - l_t} &= (1 - \alpha)\frac{y_{et}}{l_t} \\ \eta &= (1 - \alpha)\frac{y_{et}}{c_{et}}\frac{1 - l_t}{l_t} \\ &= (1 - \alpha)\frac{Y_t}{C_t}\frac{1 - l_t}{l_t}. \end{aligned}$$

Given the knowledge of $\alpha$, Cooley and Prescott (1995) calibrate $\eta$ using this equation. For $l_t$, they use micro evidence from time allocation studies by Ghez and Becker (1975) and Juster and Stafford (1991) that shows one third of their discretionary time is allocated to market activities.

- Final task is to run experiment. For this purpose, we need to compute an equilibrium. Note that our economy is summarized by 4 equations with suitable boundary conditions

$$
\begin{aligned}
\frac{\eta c_{et}}{1-l_t} &= (1-\alpha)\frac{y_{et}}{l_t} \\
\frac{1}{c_{et}} &= \int \frac{\beta}{c_{et+1}} \frac{\left[\alpha \frac{y_{et+1}}{k_{et+1}} + 1 - \delta\right]}{1+g} dQ\left(\varepsilon_{t+1}\right) \\
k_{et+1} &= \frac{y_{et} + (1-\delta) k_{et} - c_{et}}{(1+g)(1+n)}, \ \ given \ k_{e0} \\
\ln z_{t+1} &= \zeta \ln z_t + \varepsilon_{t+1}, \ \ given \ z_0 \ \varepsilon_{t+1} \tilde{} N\left(0, \sigma_\varepsilon^2\right)
\end{aligned}
$$

where $y_{et} = z_t k_{et}^{\alpha} l_t^{1-\alpha}$ and $0 = \lim_{t\to\infty} \left[\beta(1+n)\right]^t E_0 \left[\frac{\left[\alpha \frac{y_{et}}{k_{et}} + 1 - \delta\right] k_{et}}{c_{et}}\right]$.

# Numerical Analysis

- If the set of possible states of nature were small, one could find a numerical value function and policy function. The advantages of this method would be
  1. It would preserve the curvature of the model.
  2. It would allow study of the model in an equilibrium that is far from the steady sate.
- If the set of possible states of nature were large, we need to rely on a approximation of the model.

- One way to approximate model is to conduct the approximation of the derived dynamics around the steady state. This method has four step processes.
    1. To find the steady states without stochastic shocks.
    2. To construct an approximation of the model around steady state.
    3. To derive the solution of the approximation of the model.
    4. Analyze the solution.
- You can write your program to conduct this process using Matlab.

# Dynare

- Dynare can greatly help writing this program.
- What is Dynare?
    - “Dynare is a powerful and highly customizable engine, with an intuitive front-end interface, to solve, simulate and estimate DSGE models.” (Dynare User Guide)
    - “..., it is a pre-processor, and a collection of Matlab routines that has the great advantages of reading DSGE model equations written almost as in an academic paper.” (Dynare User Guide)
    - “Basically, the model and its related attributes, like a shock structure for instance, is written equation by equation in an editor of your choice. The resulting file will be called the *.mod* file. This file is the called from Matlab. This initiates for the Matlab routines used to either solve or estimate the model. ” (Dynare User Guide)

# Dynare

- **Installing Dynare**
  1. Go to
     - http://www.dynare.org/download
  2. Download Dynare 5.3 to your holder at the network center.
  3. It is also useful to look at the following site.
     - https://www.dynare.org/manual/
     - https://note.com/keisemi/n/naf5c33c8844d

- **The Procedure:**
  1. You write a *filename.mod* file, which contains the essential information on your model. Then, using this information, Dynare constructs a *filename.m* file that Matlab can then run.
  2. Save your *filename.mod* file somewhere that Matlab can find it.
  3. In Matlab,
     1. Set a path to Dynare folder including sub-folder.
     2. Change the directory to the folder that contains *filename.mod*.
     3. Type

`dynare` *filename*

- **Basic Instructions for the program by Dynare**.
  1. Each instruction must be terminated by a semicolon, ;.
  2. You can comment out any line by starting the line with two forward slashes, //, or comment out an entire section by starting with /* and ending with */.

# Dynare

- The program contains the following 4 blocks.
  1. **Preamble:** lists variables and parameters.
  2. **Declaration of the model:** spells out the model.
  3. **Set initial conditions:** Specify initial conditions and gives indication to find the steady state of a model.
  4. **Specifying Shocks:** defines the shocks to the system.
  5. **Computation:** instructs Dynare to undertake specific operations (e.g. forecasting, estimating impulse response functions, $\frac{dx_{t+j}}{d\varepsilon_t}$ where $x_{t+j} = c_{et+j}$, $y_{et+j}$, $k_{et+j}$, $l_{t+j}$ and $\ln z_{t+j}$, and providing various descriptive statistics.)

- **Preamble** consists of the some setup information: the endogenous and exogenous variables, the parameters and assign values to these parameters.
  1. **var** specifies the lists of endogenous variables, to be separated by commas. In our example, endogenous variables are detrended consumption per capita $c_e$, detrended capital per capita $k_e$, detrended GNP per capita ($y_e$), hours of work ($l$), and the technology ($z$).
  2. **varexo** specifies the list of exogenous variables that will be shocked. In our example, an exogenous variable is a stochastic term, $\varepsilon$.
  3. **parameters** specifies the list of parameters of the model and assigns values:

# Dynare

```
var c, k, l, y, lnz;
varexo epsilon;
parameters beta, alpha, eta, delta, g, n, zeta;
beta=0.987; alpha=0.4; eta=1.78; delta=0.012; g=0.0156; n=0.012;
zeta=0.95;
```

- **Declaration of the model:** Now it is time to specify the model itself. You must write

```
model;
describing your model;
end;
```

# Dynare

- There are several rules for describing your model.

  1. There need to be as many equations as you declared endogenous variables.
  2. Time subscripts: "x" means $x_t$, "x(-n)" means $x_{t-n}$ and "x(+n)" means $x_{t+n}$.
  3. In dynare, the timing of each variable reflects when that variable is decided. Because state variables are in general determined at past, a transition equation should be written as $S_t = G(X_{t-1}, S_{t-1}, \varepsilon_t)$, but not $S_{t+1} = G(X_t, S_t, \varepsilon_{t+1})$. On the other hand, the variable shown as $x(+1)$ tells Dynare to count that variable as a forward-looking variable. These distinctions are important to insure *saddle point stability*.
  4. You can ignore the expectation terms when you write a program.

# Dynare

- Remember that our economy is summarized by the following 5 equations.

$$
\begin{aligned}
\frac{\eta c_{et}}{1-l_t} &= (1-\alpha)\frac{y_{et}}{l_t} \\
\frac{1}{c_{et}} &= \int \frac{\beta}{c_{et+1}} \frac{\left[\alpha \frac{y_{et+1}}{k_{et+1}} + 1 - \delta\right]}{1+g} dQ\left(\varepsilon_{t+1}\right) \\
k_{et+1} &= \frac{y_{et} + (1-\delta) k_{et} - c_{et}}{(1+g)(1+n)}, \ \ given\ k_{e0} \\
\ln z_{t+1} &= \zeta \ln z_t + \varepsilon_{t+1}, \ \ given\ z_0\ \ \varepsilon_{t+1} \tilde{} N\left(0, \sigma_\varepsilon^2\right) \\
y_{et} &= z_t k_{et}^{\alpha} l_t^{1-\alpha}
\end{aligned}
$$

and $0 = \lim_{t\to\infty} \left[\beta(1+n)\right]^t E_0 \left[\frac{\left[\alpha \frac{y_{et}}{k_{et}} + 1 - \delta\right] k_{et}}{c_{et}}\right]$.

# Dynare

```
model;
(eta*c)/(1-l)=(1-alpha)*(y/l);
1/c=(beta/c(+1))*((alpha*(y(+1)/k)+1-delta)/(1+g));
k=(y+(1-delta)*k(-1)-c)/((1+g)*(1+n));
lnz=zeta*lnz(-1)+epsilon;
y=exp(lnz)*(k(-1)^alpha)*(l^(1-alpha));
end;
```

- **Set initial conditions:** Next, you must tell Dynare initial conditions.

initval;
Listing the initial conditions
end;

- Dynare can help finding your model's steady sate by calling the appropriate Matlab functions. But it is usually only successful if the initial values you entered are closed to the true steady state. Hence it is good idea that you can solve your steady state and find the steady state values by yourself.

- **Assignment:** Compute the steady state value of $(c_e, y_e, k_e, l)$ when $\beta = 0.987$, $\alpha = 0.4$, $\eta = 1.78$, $\delta = 0.012$, $g = 0.0156$, $n = 0.012$, and $\zeta = 0.95$ using Matlab. You have two ways of doing so. Choose one of them.
  1. Derive an analytical solution of $(c_e, y_e, k_e, l)$ on the steady state when $\sigma_\varepsilon^2 = 0$. Then compute $(c_e, y_e, k_e, l)$ using Matlab or Excel.
  2. Use the command 'fsolve' and 'function' in Matlab to find the steady state value of $(c_e, y_e, k_e, l)$ when $\sigma_\varepsilon^2 = 0$.

# Dynare

- In addition, if you start your analysis from your steady state, you can add

```
steady;
```

- Then Dynare recognizes that your initial conditions are just approximations and you wish to start your analysis from your steady sate.

# Dynare

*initval;*
*k=15.84207; c=0.992578; l=0.35531; y=1.622889; lnz=0;*
*epsilon=0;*
*end;*
*steady;*

- **Adding Shocks:** We then specify the innovations and their matrix of variance–covariance. This is done by writing:

```
shocks;
var [the name of stochastic term] =[number];
end;
```

- In our program,

*shocks;*
*var epsilon=0.0007^2;*
*end;*

- **Computation:** Finally, you tell Dynare that you are done by typing

```
stoch_simul;
```

- This commands instructs Dynare to compute a Taylor approximation of the decision and transition function for the model, impulse response functions, $\frac{dx_{t+j}}{d\varepsilon_t}$ where $x_{t+j} = c_{et+j}$, $y_{et+j}$, $k_{et+j}$, $l_{t+j}$ and $\ln z_{t+j}$, and various descriptive statistics (moments, variance decomposition, correlation and autocorrelation coefficients.)
- There are several options which you can learn later. For example, by default, Dynare drops the first 100 values, but you can change that using the appropriate option.

# Dynare

## Policy and Transition Functions

| | c | k | l | y | lnz |
|---|---|---|---|---|---|
| Constant | 0.992562 | 15.8421 | 0.355315 | 1.622905 | 0 |
| (correction) | -1.5E-05 | 0.00003 | 0.000006 | 0.000016 | 0 |
| k(-1) | 0.036802 | 0.953569 | -0.00442 | 0.028868 | 0 |
| lnz(-1) | 0.353825 | 1.747009 | 0.221722 | 2.149378 | 0.95 |
| epsilon | 0.372447 | 1.838957 | 0.233391 | 2.262503 | 1 |
| k(-1),k(-1) | -0.0004 | -0.00039 | 0.000114 | -0.0008 | 0 |
| lnz(-1),k(-1) | 0.007611 | 0.04193 | 0.001794 | 0.050706 | 0 |
| lnz(-1),lnz(-1) | 0.122127 | 1.040961 | -0.01523 | 1.192013 | 0 |
| epsilon,epsilon | 0.13532 | 1.15342 | -0.01687 | 1.32079 | 0 |
| k(-1),epsilon | 0.008011 | 0.044137 | 0.001888 | 0.053374 | 0 |
| lnz(-1),epsilon | 0.257108 | 2.191497 | -0.03206 | 2.509501 | 0 |

## Theoretical Moments

| VARIABLE | MEAN | STD. DEV. | VARIANCE |
|---|---|---|---|
| c | 0.9932 | 0.029 | 0.0008 |
| k | 15.8609 | 0.616 | 0.3795 |
| l | 0.3553 | 0.0039 | 0 |
| y | 1.6243 | 0.0641 | 0.0041 |
| lnz | 0 | 0.0224 | 0.0005 |

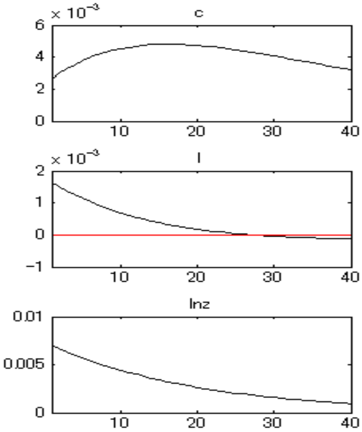
## Matrix of Correlations

| Variables | c | k | I | y | lnz |
|---|---|---|---|---|---|
| c | 1 | 0.9865 | 0.4094 | 0.9176 | 0.8168 |
| k | 0.9865 | 1 | 0.2542 | 0.8399 | 0.7111 |
| I | 0.4094 | 0.2542 | 1 | 0.7384 | 0.8608 |
| y | 0.9176 | 0.8399 | 0.7384 | 1 | 0.9789 |
| lnz | 0.8168 | 0.7111 | 0.8608 | 0.9789 | 1 |

## Coefficient of Autocorrelation

| Order | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| c | 0.9955 | 0.9891 | 0.981 | 0.9714 | 0.9605 |
| k | 0.9988 | 0.9954 | 0.9899 | 0.9827 | 0.9739 |
| l | 0.9097 | 0.8258 | 0.7479 | 0.6756 | 0.6085 |
| y | 0.969 | 0.9386 | 0.909 | 0.88 | 0.8517 |
| lnz | 0.95 | 0.9025 | 0.8574 | 0.8145 | 0.7738 |

# Dynare




Implus Response Functions

- **Assignment:** Reproduce what I did by using Dynare.

- **Linear Approximation**: Let me briefly illustrate how dynare conducts linearization. We just discuss the first order approximation, though Dynare can conduct the second order approximation. The spirits of derivation for the second order approximation is the same as the first order one.
- Note that DSGE model is in general summarized by the collection of the first order equilibrium conditions that take the general form:

$$E_t \left[ f \left( x_{t+1}, x_t, x_{t-1}, u_t \right) \right] = 0$$

where $x_t$ is the vector of endogenous variables and $u_t$ is the vector of stochastic shocks.

- In our example, $x_t = [c_{et}, k_{et}, l_{et}, y_{et}, \ln z_t]$ and $u_t = \varepsilon_t$, and

$$f\left(x_{t+1}, x_t, x_{t-1}, u_t\right) = \begin{bmatrix} \frac{\eta c_{et}}{1-l_t} - (1-\alpha)\frac{y_{et}}{l_t} \\ \frac{1}{c_{et}} - \frac{\beta}{c_{et+1}} \frac{\left[\alpha \frac{y_{et+1}}{k_{et}} + 1 - \delta\right]}{1+g} \\ k_{et} - \frac{y_{et} + (1-\delta)k_{et-1} - c_{et}}{(1+g)(1+n)} \\ \ln z_t - \zeta \ln z_{t-1} - \varepsilon_t \\ y_{et} - \exp\left(\ln z_t\right) k_{et-1}^{\alpha} l_t^{1-\alpha} \end{bmatrix}$$

# Dynare

## Lemma

*Suppose that* $\Gamma(x)$ *is* $n \times 1$ *vector and* $x$ *is* $m \times 1$ *vector. Then*

$$\Gamma(x) \approx \Gamma(x^*) + \nabla\Gamma(x^*)(x - x^*)$$

*where*

$$\nabla\Gamma(x^*) = \begin{bmatrix} \frac{\partial\Gamma^1(x^*)}{\partial x_1} & .. & \frac{\partial\Gamma^1(x^*)}{\partial x_m} \\ .. & & .. \\ \frac{\partial\Gamma^n(x^*)}{\partial x_1} & .. & \frac{\partial\Gamma^n(x^*)}{\partial x_m} \end{bmatrix}$$

*and* $\Gamma^i(x^*)$ *and* $x_j$ *are the* $i$*th element of* $\Gamma(x^*)$ *and the* $j$*th element of* $x$ *respectively.*

# Dynare

- Applying the corollary,

$$\begin{aligned} & f\left(x_{t+1}, x_t, x_{t-1}, u_t\right) \\ = \;& f\left(x^*, x^*, x^*, 0\right) + \left[M_+, M_0, M_-, M_u\right]\left[\hat{x}_{t+1}, \hat{x}_t, \hat{x}_{t-1}, \hat{u}_t\right]^T \\ = \;& f\left(x^*, x^*, x^*, 0\right) + M_+\hat{x}_{t+1} + M_0\hat{x}_t + M_-\hat{x}_{t-1} + M_u\hat{u}_t \end{aligned}$$

where $\hat{x} = x - x^*$, $M_+$, $M_0$, $M_-$ and $M_u$ are the Matrix with the element of $\frac{\partial\Gamma(y)}{\partial y}$ where $y$ is $x_{t+1}$, $x_t$, $x_{t-1}$ and $u_t$ evaluated at the steady state. Note that on the steady sate

$$f\left(x^*, x^*, x^*, 0\right) = 0$$

and by definition

$$E_t\left[f\left(x_{t+1}, x_t, x_{t-1}, u_t\right)\right] = 0$$

- Hence, after the first order approximation, the dynamics of DSGE model around the steady state is summarized by

$$0 = E_t\left[M_+\hat{x}_{t+1} + M_0\hat{x}_t + M_-\hat{x}_{t-1} + M_u\hat{u}_t\right]$$

- **Solution to the linearized dynamics:** Because the dynamics of DSGE model is linear around steady state, we seek to obtain the following solution.

$$\hat{x}_t = P_x \hat{x}_{t-1} + P_u \hat{u}_t$$

and the dynamics is stable.

- Note that this solution implies

$$\begin{aligned} \hat{x}_{t+1} &= P_x \hat{x}_t + P_u \hat{u}_{t+1} \\ &= P_x \left[ P_x \hat{x}_{t-1} + P_u \hat{u}_t \right] + P_u \hat{u}_{t+1} \end{aligned}$$

# Dynare

- Hence,

$$
\begin{aligned}
0 &= E_t\left[M_+\hat{x}_{t+1} + M_0\hat{x}_t + M_-\hat{x}_{t-1} + M_u\hat{u}_t\right] \\
&= E_t\left[\begin{array}{c} M_+\left[P_x\left[P_x\hat{x}_{t-1} + P_u\hat{u}_t\right] + P_u\hat{u}_{t+1}\right] \\ +M_0\left[P_x\hat{x}_{t-1} + P_u\hat{u}_t\right] + M_-\hat{x}_{t-1} + M_u\hat{u}_t \end{array}\right] \\
&= \left[M_+P_x^2 + M_0P_x + M_-\right]\hat{x}_{t-1} \\
&\quad + \left\{\left[M_+P_x + M_0\right]P_u + M_u\right\}\hat{u}_t + M_+P_uE_t\left[u_{t+1}\right] \\
&= \left[M_+P_x^2 + M_0P_x + M_-\right]\hat{x}_{t-1} + \left\{\left[M_+P_x + M_0\right]P_u + M_u\right\}\hat{u}_t
\end{aligned}
$$

for any $\hat{x}_{t-1}$ and $\hat{u}_t$.

# Dynare

- Hence, $P_x$ and $P_u$ must be chosen to satisfy

$$
\begin{aligned}
0 &= M_+ P_x^2 + M_0 P_x + M_- \\
0 &= \left[M_+ P_x + M_0\right] P_u + M_u
\end{aligned}
$$

- Dynare computes $P_x$ and $P_u$ to satisfy this relationship and to insure the stability of dynamics.

# Debate on How to Identify Parameters and the Source of Shocks

- Woodford (2009): "While macroeconomics is often thought of as a deeply divided field, with less of a shared core and correspondingly less cumulative progress than other areas of economics, in fact, there are fewer fundamental disagreements among macroeconomists now than in past decades."
  1. "it is now widely agreed that macroeconomic analysis should employ models with coherent intertemporal general-equilibrium foundations."
  2. "it is also widely agreed that it is desirable to base quantitative policy analysis on econometrically validated structural models."
     - "A variety of empirical methods are used, both for data characterization and for estimation of structural relations, and researchers differ in their taste for specific methods, often depending on their willingness to employ methods that involve more specific a priori assumptions. But the existence of such debates should not conceal the broad agreement on more basic issues of method. Both "calibrationists" and the practitioners of Bayesian estimation of DSGE models agree on the importance of doing "quantitative theory."

# Debate on How to Identify Parameters and the Source of Shocks

- Chari, Kehoe, and McGrattan (2009): " Macroeconomists have largely converged on method, model design, reduced-form shocks, and principles of policy advice. Our main disagreements today are about implementing the methodology."
  1. "The tradition favored by many neoclassicals (including us) is to keep a macro model simple, keep the number of its parameters small and well motivated by micro facts, and put up with the reality that no model can, or should, fit most aspects of the data. Recognize, instead, that a small macro model consistent with the micro data can still be useful in clarifying how to think about policy. "
  2. "The competing tradition is favored by many New Keynesians. ...this tradition emphasizes the need for macro models to fit macro data well. The urge to improve the macro fit leads researchers in this tradition to add many shocks and other features to their models and, then, to use the same old aggregate data to estimate the associated new parameters. This tradition does not include the discipline of microeconomic evidence. "

# Debate on How to Identify Parameters and the Source of Shocks

- Woodford (2009): "While the study of business fluctuations is no longer driven by the kind of disagreements about the foundations of macroeconomic analysis that characterized the decades following World War II, important differences in methodological orientation remain among macroeconomists. Probably the most obvious divisions concern the importance attached, by different researchers, to work aspiring to "pure science" relative to work intended to address applied problems."
  - "Some protest that the current generation of empirical DSGE models, mentioned above as illustrations of the new synthesis in methodology, have not been validated with sufficiently rigorous methods to be used in policy analysis (e.g., Chari, Kehoe, and Ellen R. McGrattan, 2008). Proponents of this view do not typically assert that some other available model would be more reliable for that purpose. Instead, they argue that scholars with intellectual integrity have no business commenting on policy issues."

# Discrete Choice and Labor Search

- So far, we presume that the first order conditions are valid. This assumption may not be suitable when the choice set is discrete.
- There are several discrete decisions in our life: to marry with somebody, to accept a job, to move a different region and so on.
- In particular, unemployment is likely to be influenced by the behavior of job search.
- We discuss how discrete choice model can be analyzed in a framework of a job search model.

- **Original Problem:**

$$\max_{\{d_t\}_{t=0}^{\infty}} E_\tau \sum_{t=\tau}^{\infty} \beta^{(t-\tau)} \left[ I\left(s_t = e\right) w_t + I\left(s_t = u\right) b \right]$$

$$\begin{aligned} s.t.\ s_{t+1} &= G\left[s_t, o_t, d_t\right], \\ w_{t+1} &= \Omega\left[s_t, o_t, d_t, w_t, \varepsilon_t\right] \end{aligned}$$

where $w_t$ is wage $b$ is the unemployment benefits or/and the value of leisure, $I\left(s_t = x\right) = 1$ if $s_t = x$, $I\left(s_t = x\right) = 0$ otherwise, and

1. $s_t \in \{e, u\}$, $e$ and $u$ indicate employment and unemployment, respectively.
2. $o_t \in \{m, n\}$, $m$ and $n$ indicate meeting and not meeting with a job, respectively
3. $d_t \in \{a, r\}$, $a$ and $r$ indicate accepting and rejecting an offer, respectively.

- We assume the following probability distribution

$$\begin{aligned} \Pr\left(\varepsilon_t | o_t = m\right) &= Q_\varepsilon\left(\varepsilon_t\right), \\ \Pr\left(\varepsilon_t = 0 | o_t = n\right) &= 1, \\ \Pr\left(\varepsilon_t = x | o_t = n\right) &= 0, \ \textit{for any } x > 0 \end{aligned}$$

$$\begin{aligned} \Pr\left(o_t = m | s_t = e\right) &= 0, \ \Pr\left(o_t = n | s_t = e\right) = 1, \\ \Pr\left(o_t = m | s_t = u\right) &= \lambda, \ \ \Pr\left(o_t = n | s_t = u\right) = 1 - \lambda \end{aligned}$$

- We also assume the following properties of the transition functions:

$$\begin{aligned} e &= G\left[e, o_t, d_t\right], u = G\left[u, n, d_t\right] = G\left[u, m, r\right], e = G\left[u, m, a\right], \\ w &= \Omega\left[e, o_t, d_t, w, \varepsilon_t\right], \ 0 = \Omega\left[u, n, d_t, w, \varepsilon_t\right] = \Omega\left[u, m, r, w, \varepsilon_t\right], \\ \varepsilon_t &= \Omega\left[u, m, a, w, \varepsilon_t\right] \end{aligned}$$

- Bellman Equation

$$\begin{aligned} V(e,w) &= w + \beta V\left(G\left[e,n,d_t\right], \Omega\left[e,n,d_t,w,\varepsilon_t\right]\right) \\ &= w + \beta V(e,w) \end{aligned}$$

$$\begin{aligned} & V(u,w) \\ = \;& b + \beta \left[ \begin{array}{c} \lambda \int \max \left\{ \begin{array}{c} V\left(G\left[u,m,a\right], \Omega\left[u,m,a,w,\varepsilon\right]\right), \\ V\left(G\left[u,m,r\right], \Omega\left[u,m,r,w,\varepsilon\right]\right) \end{array} \right\} dQ(\varepsilon) \\ + (1-\lambda) V\left(G\left[u,n,d_t\right], \Omega\left[u,n,d_t,w,\varepsilon\right]\right) \end{array} \right] \\ = \;& b + \beta \left[ \lambda \int \max\left\{V(e,\varepsilon), V(u,0)\right\} dQ(\varepsilon) + (1-\lambda) V(u,0) \right] \end{aligned}$$

for any $w \geq 0$. Hence, $V(u,0) = V(u,w)$.

# Discrete Choice and Labor Search

- Define

$$U \equiv V(u,0), W(w) \equiv V(e,w)$$

- We can rewrite Bellman equation by

$$W(w) = w + \beta W(w)$$

$$\begin{aligned} U &= b + \beta\left[\lambda \int \max\{W(\varepsilon), U\}\, dQ(\varepsilon) + (1-\lambda) U\right] \\ &= b + \beta\left[\lambda \int \max\{W(\varepsilon) - U, 0\}\, dQ(\varepsilon) + U\right] \\ (1-\beta) U &= b + \beta\lambda \int \max\{W(\varepsilon) - U, 0\}\, dQ(\varepsilon) \end{aligned}$$

- Note that $W(w)$ can be solved as

$$W(w) = \frac{w}{1-\beta}$$

- Hence

$$W'(w) > 0$$

- **Reservation wage:** Strictly increasing function of $W(\cdot)$ means that there exists a unique reservation wage $w_R$ that has a property

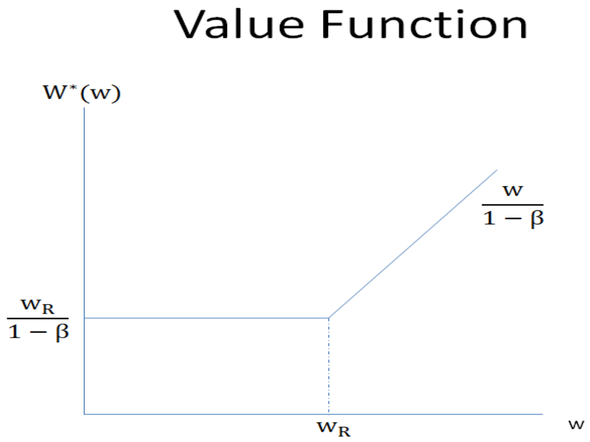
$$\begin{aligned} W(w_R) &= U \\ W(w) &> U, \ \forall w > w_R \\ W(w) &< U, \ \forall w < w_R \end{aligned}$$

- **Optimal Decision:** an optimal decision is

$$\begin{aligned} d_t &= a, \ w \geq w_R \\ &= r, \ w < w_R \end{aligned}$$

- **Value Function**: Define $W^*(w) = \max[W(w), U]$. Then

$$
\begin{aligned}
W^*(w) &= \max[W(w), W(w_R)] \\
&= \frac{w}{1-\beta} \; \text{if } w \geq w_R \\
&= \frac{w_R}{1-\beta} \; \text{if } w < w_R
\end{aligned}
$$


Value Function
$W^*(w)$
$\frac{w_R}{1-\beta}$
$\frac{w}{1-\beta}$
$w_R$
$w$

- **Reservation wage:**

$$\begin{aligned}
(1-\beta) W(w_R) &= b + \beta\lambda \int \max\left\{W(\varepsilon) - W(w_R), 0\right\} dQ(\varepsilon) \\
w_R &= b + \beta\lambda \int \max\left\{\frac{\varepsilon}{1-\beta} - \frac{w_R}{1-\beta}, 0\right\} dQ(\varepsilon) \\
w_R &= b + \frac{\beta\lambda}{1-\beta} \int \max\left\{\varepsilon - w_R, 0\right\} dQ(\varepsilon) \\
&= b + \frac{\beta\lambda}{1-\beta} \int_{w_R} \left[\varepsilon - w_R\right] dQ(\varepsilon) > b
\end{aligned}$$

where $\frac{\beta\lambda}{1-\beta} \int_{w_R} \left[\varepsilon - w_R\right] dQ(\varepsilon)$ is the option value to wait.

**Lemma**

*Suppose that* $Q(\bar{w}) = 1$. *If* $\bar{w} > b$, *there exists a unique* $w_R$ *that satisfies*

$$w_R = b + \frac{\beta\lambda}{1-\beta} \int_{w_R} [\varepsilon - w_R] \, dQ(\varepsilon)$$

- **Proof:** Define

$$T\left(w_R : b\right) : T\left(w_R : b\right) = w_R - \left[b + \frac{\beta\lambda}{1-\beta}\int_{w_R}\left[\varepsilon - w_R\right] dQ\left(\varepsilon\right)\right]$$

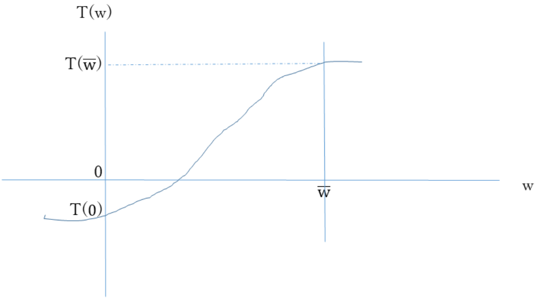
Note that $T$ function has a property:

$$\begin{aligned} T_1\left(w_R : b\right) &= 1 - \frac{\beta\lambda}{1-\beta}\left\{-\left(w_R - w_R\right) Q'\left(w_R\right) - \int_{w_R} dQ\left(\varepsilon\right)\right\} \\ &= 1 + \frac{\beta\lambda}{1-\beta}\left[\int_{w_R} dQ\left(\varepsilon\right)\right] > 0 \end{aligned}$$

$$T\left(0 : b\right) = -\left[b + \frac{\beta\lambda}{1-\beta}\int_0 \varepsilon dQ\left(\varepsilon\right)\right] < 0, \ \ T\left(\bar{w} : b\right) = \bar{w} - b > 0$$

Because $T\left(w_R : b\right)$ is continuous in $w_R$, there exists a unique $w_R$ that satisfies $T\left(w_R : b\right) = 0$.

point




$_2 6.pdf$

- $\frac{dw_R}{db}$: Note that

$$T_2\left(w_R : b\right) < 0$$

Because $T\left(w_R : b\right) = 0$ for any pair of $w_R$ and $b$,

$$\begin{aligned} 0 &= T_1\left(w_R : b\right) dw_R + T_2\left(w_R : b\right) db \\ \frac{dw_R}{db} &= -\frac{T_2\left(w_R : b\right)}{T_1\left(w_R : b\right)} > 0 \end{aligned}$$

- That is, increases in unemployment benefits raise workers' reservation wages.

- **Hazard Rate:** the probability of an unemployed worker accepting a job at date $t$ is called the hazard rate.

$$H = \Pr(s_t = e | s_s = u, \forall s < t) = \lambda [1 - Q(w_R)]$$

- Using hazard rate, the probability of being unemployed for exactly $t$ periods can be expressed as follows (we assume $\Pr(s_0 = u) = 1$).

$$\begin{aligned}
& \Pr(\{s_t = e\} \cap \{s_s = u, \forall s < t\}) \\
= & \Pr(s_t = e | s_s = u, \forall s < t) \Pr(s_s = u, \forall s < t) \\
= & \Pr(s_t = e | s_s = u, \forall s < t) \\
& \Pr(s_{t-1} = u | s_s = u, \forall s < t-1) \Pr(s_s = u, \forall s < t-1) \\
= & \Pr(s_t = e | s_s = u, \forall s < t) \\
& \prod_{i=2}^{t-1} \Pr(s_{i-1} = u | s_s = u, \forall s < i-1) \Pr(s_0 = u) \\
= & (1 - H)^{t-1} H
\end{aligned}$$

- Therefore, the expected duration of unemployment is

$$E[t] = \sum_{t=1}^{\infty} t(1-H)^{t-1} H$$

**Lemma**

*Assume that $H \in (0,1)$. The expected duration of unemployment is the inverse of hazard rate:*

$$E[t] = \frac{1}{H}$$

- **Proof:** Note that for all $H$

$$1 = \sum_{t=1}^{\infty} (1 - H)^{t-1} H$$

Hence

$$\begin{aligned} 0 &= \frac{d \sum_{t=1}^{\infty} (1 - H)^{t-1} H}{dH} \\ &= \sum_{t=1}^{\infty} (1 - H)^{t-1} - \sum_{t=1}^{\infty} (t - 1) (1 - H)^{t-2} H \\ &= \frac{(1 - H) \sum_{t=1}^{\infty} (1 - H)^{t-1} - \sum_{t=1}^{\infty} (t - 1) (1 - H)^{t-1} H}{1 - H} \end{aligned}$$

- **Proof:**

$$\begin{aligned}
\sum_{t=1}^{\infty}(t-1)(1-H)^{t-1}H &= (1-H)\sum_{t=1}^{\infty}(1-H)^{t-1} \\
\sum_{t=1}^{\infty}t(1-H)^{t-1}H &= \sum_{t=1}^{\infty}(1-H)^{t-1} \\
&= \frac{\sum_{t=1}^{\infty}(1-H)^{t-1}H}{H} \\
&= \frac{1}{H}
\end{aligned}$$

- $\frac{dE[t]}{db}$

$$E[t] = \frac{1}{\lambda [1 - Q(w_R)]}$$

  Because $\frac{dw_R}{db} > 0$, $\frac{dE[t]}{db} > 0$.

- Because an increase in unemployment benefits raise the reservation value of workers, workers are more likely to reject offers. Therefore, the expected duration of unemployment is longer.

# Stochastic continuous DP and Labor Search

- In general, it is not easy to analyze stochastic process under continuous time model.
- However, if we assume a particular process such as a Poisson process or Brownian motion, we can derive a handy solution.
- Researchers often analyze a continuous search model under the assumption that the arrivals of a job offer follow a Poisson process.
- Using the similar model, we discuss how we can analyze continuous model.

- Consider a possibility of multiple offers

$$
\begin{aligned}
U &= b+\beta\left[\begin{array}{c}\sum_{n=1}^{\infty}\lambda(n)\int\max\{W(\omega_n),U\}\,dQ(\omega_n:n)\\ +\left[1-\sum_{n=1}^{\infty}\lambda(n)\right]U\end{array}\right]\\
&= b+\beta\left[\sum_{n=1}^{\infty}\lambda(n)\int\max\{W(\omega_n)-U,0\}\,dQ(\omega_n:n)+U\right]
\end{aligned}
$$

$$
(1-\beta)U = b+\beta\sum_{n=1}^{\infty}\lambda(n)\int\max\{W(\omega_n)-U,0\}\,dQ(\omega_n:n)
$$

where $\omega_n = \max\{\varepsilon_1, \varepsilon_2, ..\varepsilon_n\}$.

- Assume that $\beta = \frac{1}{1+\Delta\varrho}$ and replace $w$, $b$ and $\lambda(n)$ by $\Delta w$, $\Delta b$ and $\lambda(n, \Delta)$. We can rewrite the discrete problem as follows.

$$W(w) = \Delta w + \frac{1}{1+\Delta\varrho} W(w)$$

$$\begin{aligned} & \left(1 - \frac{1}{1+\Delta\varrho}\right) U \\ = \; & \Delta b + \frac{1}{1+\Delta\varrho}\left[\sum_{n=1}^{\infty} \lambda(n, \Delta) \int \max\{W(\omega_n) - U, 0\}\, dQ(\omega_n : n)\right] \end{aligned}$$

- $W(w)$

$$\begin{aligned} W(w) &= \Delta w + \frac{1}{1+\Delta\varrho} W(w) \\ (1+\Delta\varrho) W(w) &= (1+\Delta\varrho)\Delta w + W(w) \\ \Delta\varrho W(w) &= \Delta w + \Delta^2 \varrho w \\ \varrho W(w) &= w + \Delta\varrho w \end{aligned}$$

Hence

$$W(w) = \frac{w}{\varrho}$$

# Stochastic continuous DP and Labor Search

- $U$

$$\begin{aligned}&\left(1-\frac{1}{1+\Delta\varrho}\right)U\\ = \quad &\Delta b+\frac{\sum_{n=1}^{\infty}\lambda\left(n,\Delta\right)\int\max\left\{W\left(\omega_n\right)-U,0\right\}dQ\left(\omega_n:n\right)}{1+\Delta\varrho}\end{aligned}$$

$$\begin{aligned}\Delta\varrho U &= (1+\Delta\varrho)\,\Delta b \\ &\quad+\sum_{n=1}^{\infty}\lambda\left(n,\Delta\right)\int\max\left\{W\left(\omega_n\right)-U,0\right\}dQ\left(\omega_n:n\right) \\ &= \Delta b+\Delta^2\varrho b \\ &\quad+\sum_{n=1}^{\infty}\lambda\left(n,\Delta\right)\int\max\left\{W\left(\omega_n\right)-U,0\right\}dQ\left(\omega_n:n\right) \\ \varrho U &= b+\Delta\varrho b+\sum_{n=1}^{\infty}\frac{\lambda\left(n,\Delta\right)}{\Delta}\int\max\left\{W\left(\omega_n\right)-U,0\right\}dQ\left(\omega_n:n\right)\end{aligned}$$

- Assume that $\lambda(n, \Delta)$ is given by a Poisson density function with parameter $\lambda\Delta$:

$$\lambda(n, \Delta) = \frac{(\lambda\Delta)^n e^{-\lambda\Delta}}{n!}$$

Note that

$$\begin{aligned} \lim_{\Delta \to 0} \frac{\lambda(n, \Delta)}{\Delta} &= \lim_{\Delta \to 0} \frac{\lambda^n \Delta^{n-1} e^{-\lambda\Delta}}{n!} \\ &= \lambda \text{ if } n = 1 \\ &= 0 \text{ if } n \geq 2 \end{aligned}$$

- This limiting case is called a Poisson process.

# Stochastic continuous DP and Labor Search

- Hence, the continuous version of Bellman equation is

$$\begin{aligned}\varrho U &= b + \lambda \int \max \{W(\omega_1) - U, 0\} \, dQ(\omega_1 : 1) \\ &= b + \lambda \int \max \{W(\varepsilon) - U, 0\} \, dQ(\varepsilon)\end{aligned}$$

- **Assignment:**
  1. Suppose that $W^*(w) = \max\{W(w), U\}$. Draw the picture of the value function $W^*(w)$.

- Because $W(w) = \frac{w}{\varrho}$,

$$\varrho U = b + \lambda \int \max\left\{\frac{\varepsilon}{\varrho} - U, 0\right\} dQ(\varepsilon)$$

- Hence, there exists a reservation wage $w_R$

$$\frac{w_R}{\varrho} = U.$$

and it must satisfy

$$\begin{aligned} w_R &= b + \lambda \int \max\left\{\frac{\varepsilon}{\varrho} - \frac{w_R}{\varrho}, 0\right\} dQ(\varepsilon) \\ &= b + \frac{\lambda}{\varrho} \int \max\{\varepsilon - w_R, 0\} dQ(\varepsilon) \\ &= b + \frac{\lambda}{\varrho} \int_{w_R} (\varepsilon - w_R) dQ(\varepsilon) > b \end{aligned}$$

- **Assignment:**
  1. Show that If $\bar{w} > b$ where $Q(\bar{w}) = 1$, there exist a unique $w_R$ that satisfies
  $$w_R = b + \frac{\lambda}{\varrho} \int_{w_R} [\varepsilon - w_R] \, dQ(\varepsilon)$$
  2. Show that $\frac{dw_R}{db} > 0$.

- **Hazard Rate**: Similar to the discrete time model, hazard rate can be obtained as follows.

$$H = \Pr\left(s_t = e | s_s = u, \forall s < t\right) = \lambda\left[1 - Q\left(w_R\right)\right]$$

- Let us assume that $F(t)$ is the probability to accept before at date $t$. Then the hazard rate can be written as

$$\begin{aligned}\Pr\left(s_t = e | s_s = u, \forall s < t\right) &= \frac{\Pr\left(\{s_t = e\} \cap \{s_s = u, \forall s < t\}\right)}{\Pr\left(s_s = u, \forall s < t\right)} \\ &= \frac{F'(t)}{1 - F(t)}\end{aligned}$$

It means that

$$\begin{aligned}F'(t) &= H\left[1 - F(t)\right] \\ F'(t) + HF(t) &= H\end{aligned}$$

# Stochastic continuous DP and Labor Search

.

- Solve the differential Equation:

$$
\begin{aligned}
H &= F'(t) + HF(t) \\
He^{Ht} &= F'(t)e^{Ht} + HF(t)e^{Ht} \\
&= \frac{dF(t)e^{Ht}}{dt} \\
\int_0^T He^{Ht} dt &= \left[F(t)e^{Ht}\right]_0^T \\
\left[e^{Ht}\right]_0^T &= F(T)e^{HT} - F(0) \\
e^{HT} - 1 &= F(T)e^{HT} \\
F(T) &= e^{-HT}\left[e^{HT} - 1\right] = 1 - e^{-HT}
\end{aligned}
$$

.

- Hence, a probability that a worker stay unemployment pool exactly $t$ period is

$$\Pr\left(\{s_t = e\} \cap \{s_s = u, \forall s < t\}\right) = F'(t) = He^{-Ht}$$

## Lemma

$$E[t] = \int_0^\infty tHe^{-Ht} dt = \frac{1}{H}$$

## Proof.

*Note that for any H*

$$\begin{aligned} 1 &= \int_0^\infty He^{-Ht} dt \\ 0 &= \int_0^\infty \frac{dHe^{-Ht}}{dH} dt = \int_0^\infty \left[ e^{-Ht} - tHe^{-Ht} \right] dt \end{aligned}$$

*Hence*

$$\int_0^\infty tHe^{-Ht} dt = \int_0^\infty e^{-Ht} dt = \left[ -\frac{e^{-Ht}}{H} \right]_0^\infty = \frac{1}{H}$$

□

# Equilibrium Unemployment

- So far, we demonstrate an individual search decision problem.
- In this section, we extend our analysis to the labor market equilibrium model and derive equilibrium unemployment.
- For this purpose, we must ask a question: Are the wage distribution supported in the equilibrium. When all workers are homogeneous and engage in undirected search without on the job search, the answer is 'NO'.
- This is what **the Diamond Paradox** says. We first briefly discuss the Diamond paradox.

# Equilibrium Unemployment

**Theorem**

*When all workers are homogeneous and engage in undirected search, all firms offer a reservation wage $w_R$.*

**Proof.**

*Suppose that a firm offers $w < w_R$, nobody accepts. Hence, the firm does not offer $w < w_R$. Suppose that a firm offers $w > w_R$. Because all workers accept any wage above $w_R$, the firm can offer $w' \in [w_R, w)$ and increases its profits.* □

# Equilibrium Unemployment

## Theorem

**The Diamond Paradox:** *When all workers are homogeneous and engage in undirected search, the unique equilibrium is $w = w_R = b$.*

## Proof.

*Suppose that all firms offers $w_R \geq b$. Note that the reservation wage, $w_R^*$, must satisfy*

$$\begin{aligned} w_R^* &= b + \frac{\lambda}{\varrho} \int_{w_R^*} \left(\varepsilon - w_R^*\right) dQ\left(\varepsilon\right) \\ &= b + \frac{\lambda\left(w_R - w_R^*\right)}{\varrho} \\ w_R^* &= \frac{\varrho b + \lambda w_R}{\varrho + \lambda} \leq \frac{\varrho w_R + \lambda w_R}{\varrho + \lambda} = w_R \end{aligned}$$

*where $w_R^* = w_R$ only if $w_R = b$. Because if $w_R > b$, it is not optimal for all firms to offer $w_R$, $w_R = b$ is the only possible solution.* □

# Equilibrium Unemployment

- The Diamond paradox implies there are no benefits from search. In order to derive an equilibrium unemployment, we must provide a mechanism that overcomes the Diamond paradox.
- In this lecture, we discuss two different mechanisms.
  1. Firms cannot commit posted wages and search is random. It is assumed that workers can renegotiate $w$ when firms and workers meet each other. Because of the possibility of negotiation, it is possible to have $w > b$ on the equilibrium.
  2. Firms can commit posted wages, but workers can directly search firms with preferred wage. In this case, lowering $w$ reduces the number of workers to apply the firm, which give the firm an incentive to raise $w$.
- These models are current benchmark unemployment models that can be integrated into the stochastic growth model we have studied.

# Equilibrium Unemployment

- **Search and Labor Wedge:** Recently Chari, Kehoe and McGattan (2007) argue that the efficiency and labor wedges together account for essentially all of fluctuations. Because efficiency wedges is just like a productivity, $z_t$, labor wedges captures the main deviation of data from the stochastic growth model.
  - **What is the labor wedge?** Note that our social planner problem suggests that the marginal substitution between leisure and consumption must be equal to the marginal productivity of labor.

$$\frac{\tilde{u}_2(c_{et}, 1 - l_t)}{\tilde{u}_1(c_{et}, 1 - l_t)} = z_t F_2(k_{et}, l_t)$$

    Hence any deviation of the marginal productivity of labor from the marginal substitution of leisure and consumption is considered to be labor wedge.
- Because a search model explicitly considers labor market friction, it can naturally derive this deviation.

- **Matching Function:** One of standard assumption is the existence of the matching function:

$$mN = m(uN, vN)$$

where $N$ is the number of labor force, $u$ is unemployment rate, $v$ is the vacancy rate, $m$ is the meeting rate, which shows the degree of job match per unit of time. We assume that $m(uN, vN)$ is constant returns to scale.

- Given this assumption, the Poisson arrival rate of the vacant jobs finding a worker is defined as

$$\frac{m(uN, vN)}{vN} = m\left(\frac{u}{v}, 1\right) = m\left(\frac{1}{\frac{v}{u}}, 1\right) \equiv p(\theta)$$

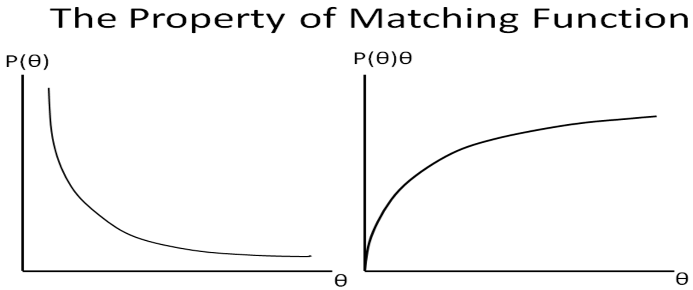
where $\theta = \frac{v}{u}$ is the measure of labor market tightness. The poisson arrival rate of an unemployed workers finding a job is

$$\frac{m(uN, vN)}{uN} = \frac{m(uN, vN)}{vN}\frac{vN}{uN} = p(\theta)\theta$$

# Equilibrium Unemployment

We assume that

$$
\begin{aligned}
p'(\theta) &< 0, p''(\theta) > 0, \frac{dp(\theta)\theta}{d\theta} > 0, \frac{d^2 p(\theta)\theta}{d^2\theta} < 0 \\
\lim_{\theta\to 0} p'(\theta) &= -\infty, \lim_{\theta\to\infty} p'(\theta) = 0, \lim_{\theta\to 0} \frac{dp(\theta)\theta}{d\theta} = \infty, \lim_{\theta\to\infty} \frac{dp(\theta)\theta}{d\theta} = 0 \\
\lim_{\theta\to 0} p(\theta) &= \infty, \lim_{\theta\to\infty} p(\theta) = 0, \lim_{\theta\to 0} p(\theta)\theta = 0, \lim_{\theta\to\infty} p(\theta)\theta = \infty
\end{aligned}
$$


The Property of Matching Function
P(θ)
θ
P(θ)θ
θ

# Equilibrium Unemployment

- **Dynamics of unemployment:**

$$\dot{u}_t = s\left[1 - u_t\right] - p\left(\theta\right)\theta u_t$$

where $s$ is the separation rate, under which a worker and a job are separated.

- **Natural Rate of Unemployment:**

$$\begin{aligned} s\left[1-u\right] &= p\left(\theta\right)\theta u \\ \left(p\left(\theta\right)\theta + s\right)u &= s \\ u^n &= \frac{s}{p\left(\theta\right)\theta + s} \end{aligned}$$

Hence, the dynamics of unemployment rate and the natural rate of unemployment is entirely influenced by labor market tightness, $\theta$. We need to develop a model that can determine $\theta$.

- **Assignment:** Assume that $\theta$ is constant over time. Solve the differential equation

$$\dot{u}_t = s\left[1 - u_t\right] - p\left(\theta\right)\theta u_t$$

and shows that the unemployment rate eventually converges to the natural rate of unemployment $u^n$.

# Equilibrium Unemployment

- Bellman equations for workers

$$\begin{aligned} \varrho W &= w + s\left[U - W\right] \\ \varrho U &= b + p\left(\theta\right)\theta\left[W - U\right] \end{aligned}$$

  where $W$ and $U$ are value functions when the worker is employed or unemployed, respectively.

- Difference from the individual decisions:
  1. The job offer rate of the previous model was $\lambda$ and constant. But, it now depends on labor market tightness $\theta$.
  2. The wage, $w$, is a unique wage determined in an equilibrium. Hence, everybody accepts the wage when the worker meets the job.
  3. There is a possibility of separation, and the probability of separation is assumed to follow poisson process with a parameter $s$.

# Equilibrium Unemployment

- Bellman equations for firms,

$$\begin{aligned} \varrho J &= mp_l - w + s\left[V - J\right] \\ \varrho V &= -\psi_s + p\left(\theta\right)\left[J - V - \psi_e\right] \end{aligned}$$

where $J$ and $V$ are the value functions when the job is occupied and vacant, respectively. The parameters, $mp_l = \max_k \left\{f\left(k\right) - rk\right\}$, $\psi_e$, $\psi_s$, denote the marginal productivity of labor, training costs (= cost for education at a firm) and search cost, respectively.

- Two remarks

1. The firms' discount rate is the same as workers' discount rate, $\varrho$ and the interest rate $r = \varrho - \delta$ is constant. The is because workers are risk neutral. To see this, consider the following problem:

$$\max_{c_t} \int_0^\infty c_t e^{-\varrho t} dt, \; s.t. \; \dot{a}_t = \rho_t a_t + w_t (1 - u_t) + b u_t - c_t, a_0 \textit{ is given}$$

Define Hamiltonian

$$H = c_t + \lambda_t \left[ \rho_t a_t + w_t (1 - u_t) + b u_t - c_t \right]$$

Therefore

$$1 = \lambda_t, \; \dot{\lambda}_t = \varrho \lambda_t - \rho_t \lambda_t \Rightarrow \dot{\lambda}_t = 0 \Rightarrow \rho_t = \varrho.$$

2. A job must incur search costs $\psi_s$, before it finds workers, but it pays training cost $\psi_e$ after it finds workers. This difference turns out to be important.

# Equilibrium Unemployment

- Free Entry Condition:

$$V = 0$$

- $J$: Free entry condition implies that $J$ must be solved by a following equation:

$$\begin{aligned} \varrho J &= mp_l - w - sJ \\ J &= \frac{mp_l - w}{\varrho + s} \end{aligned}$$

  - This condition implies that the firm's values are discounted sum of the differences between marginal productivity of labor and wage payment.
  - It is shown that this differences can be sustained when there is a specific investment. So even if household optimally chooses optimal leisure give the market wage, $w$, the marginal productivity of labor can deviate from the marginal rate of substitution between consumption and leisure: the existence of labor wedge.

# Equilibrium Unemployment

- $\theta$: Free entry condition also implies that

$$
\begin{aligned}
\psi_s &= p(\theta)\left[J - \psi_e\right] \\
\frac{\psi_s}{p(\theta)} &= J - \psi_e
\end{aligned}
$$

  - Note that the expected duration of searching workers is $\frac{1}{p(\theta)}$. Hence, $\frac{\psi_s}{p(\theta)}$ is the expected cost of search. The expected cost of search must be equal to the expected profits, $J - \psi_e$ under 0 profit conditions, which determines the tightness of labor market, $\theta$.

- **Equilibrium given wage, $w$:** Given wage $w$,
  - Firm's value function $J$ and labor market tightness $\theta$ are determined by

$$J = \frac{mp_l - w}{\varrho + s}.$$

$$\frac{\psi_s}{p(\theta)} = J - \psi_e.$$

  - Workers value functions, $W$ and $U$ are determined by

$$\varrho W = w + s[U - W].$$

$$\varrho U = b + p(\theta)\theta[W - U].$$

  - unemployment rate is determined by

$$\dot{u} = s[1 - u] - p(\theta)\theta u.$$

# Equilibrium Unemployment

- Define Surplus $S = J + W - U$. Note that from $\varrho J = mp_l - w - sJ$ and $\varrho W = w + s[U - W]$,

$$\varrho(J + W) = mp_l - s(J + W - U)$$

Hence

$$\begin{aligned}
\varrho S &= \varrho(J + W) - \varrho U \\
&= mp_l - s(J + W - U) - \varrho U \\
&= mp_l - sS - \varrho U \\
S &= \frac{mp_l - \varrho U}{\varrho + s} = \int_0^\infty [mp_l - \varrho U] e^{-[\varrho + s]t} dt
\end{aligned}$$

This means that the surplus $S$ is the discounted sum of the difference between the marginal benefits and the reservation value of unemployed workers.

# Equilibrium Unemployment

- We can also express the surplus $S$ as a function of $\theta$: $S = S(\theta)$.

$$
\begin{aligned}
S(\theta) &= \frac{mp_l - \varrho U}{\varrho + s} \\
&= \frac{mp_l - b - p(\theta)\theta[W - U]}{\varrho + s} \\
&= \frac{mp_l - b - p(\theta)\theta[J + W - U] + p(\theta)\theta J}{\varrho + s} \\
&= \frac{mp_l - b - p(\theta)\theta S(\theta) + p(\theta)\theta\left[\frac{\psi_s}{p(\theta)} + \psi_e\right]}{\varrho + s} \\
&= \frac{R(\theta)}{\varrho + s + p(\theta)\theta} = \int_0^\infty R(\theta) e^{-[\varrho + s + p(\theta)\theta]t} dt \\
R(\theta) &= mp_l - b + \theta\psi_s + p(\theta)\theta\psi_e
\end{aligned}
$$

# Equilibrium Unemployment

- Hence $S(\theta)$ is the discounted sum of the instantaneous surplus, where the instantaneous surplus, $R(\theta)$, is $mp_l - b + \theta\psi_s + p(\theta)\theta\psi_e$.
  - Note that $\theta\psi_s = \frac{vN\psi_s}{uN}$, is aggregate search costs per unemployed workers and $p(\theta)\theta\psi_e = \frac{mN\psi_e}{uN}$ is aggregate training cost per unemployed workers. Because increases in search costs and training costs raise the amount of initial investment, they reduce the number of jobs' entries and increase the amount of rent obtained after the investment.

- **Assignment:** Derive $W$ and $U$ as a function of $w$ and $\theta$.
- **Assignment:** Solve

$$w^* = \arg\max_w \frac{mp_l - w}{\varrho + s}, \ s.t. \ 0 \leq W - U,$$

and shows that $w^* = b$. (Diamond paradox).

- **Wage bargaining (Pissarides 1985):** One way to avoid the Diamond paradox is to allow workers to make a counter offer. Because a firm sunk a search cost or/and a training cost before making production, changing partner is costly. Therefore, there are some rents for them to share.

$$W + J \geq U + V = U$$

If a firm is unable to commit a particular wage, there is a room to bargain over. Pissarides (1985) assumes that wage can be determined by a bargaining.

# Wage Without Commitment

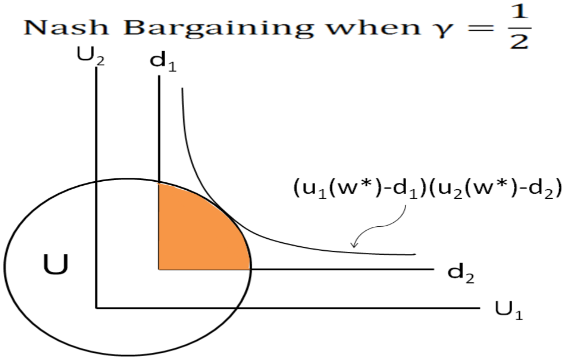
- **Generalized Nash Bargaining:** It is standard to use generalized Nash bargaining to solve this problem. Let me briefly describe what Nash bargaining is.
- **Bargaining**: Consider two players 1 and 2. They are negotiating something, say wage, $w \in W$. Define the set of all utility pairs:

$$U = \left\{ u(w) \in R^2 : u(w) = (u_1(w), u_2(w)), w \in W \right\},$$

  where $u_i$ is the ith player's utility. Define reservation utility $d = (d_1, d_2) \in D$ is utility pairs which an agent can get when they could not reach an agreement. Bargaining is defined by a function $f : U \times D \rightarrow W$.
- **Generalized Nash bargaining outcome** is the solution of the following problem

$$\begin{aligned} w^*(u, d) &= \arg\max_w (u_1(w) - d_1)^{\gamma} (u_2(w) - d_2)^{(1-\gamma)} . \\ s.t.\ u_1(w) &\geq d_1,\ u_2(w) \geq d_2,\ \gamma \in (0, 1) . \end{aligned}$$

# Wage Without Commitment



# Wage Without Commitment

- The solution of generalized Nash bargaining, $w^*$ is a unique solution which satisfies following 3 axioms.

1. **Invariance to equivalent utility representations:** If $u_i'(w) = \alpha u_i(w) + \beta$ and $d_i' = \alpha d_i + \beta$ where $i = 1$ or 2, and $\alpha > 0$, then $u_i'(w^*(u', d')) = \alpha u_i(w^*(u, d)) + \beta$. (This means that even if we change the unit of measurement, the result does not change. )
2. **Independence of irrelevant outcome:** Suppose that $U \subset U'$ and $u(w^*(u', d)) \in U$, where $u' \in U'$. Then $u(w^*(u', d)) = u(w^*(u, d))$. (This means that even if we added irrelevant alternative choices in the choice set, the result does not change.)
3. **Pareto efficiency:** If $u(w)$, $u(w') \in U$ and $u(w') > u(w)$, then $u(w^*) \neq u(w)$.

- Rubinstein (1982) shows that alternating offers bargaining model has the same solution as that of the generalized Nash bargaining.

# Wage Without Commitment

- Note that when $\gamma \to 1$ then problem is equivalent to

$$\begin{aligned} w^*(u,d) &= \arg\max_w (u_1(w) - d_1). \\ s.t.\ u_2(w) &\geq d_2. \end{aligned}$$

This is the case that a player 1 is a principal and player 2 is an agent, and that a principal makes a take-or-leave-it offer to the agent.

- On the other hand, $\gamma \to 0$, then

$$\begin{aligned} w^*(u,d) &= \arg\max_w (u_2(w) - d_2). \\ s.t.\ u_1(w) &\geq d_1. \end{aligned}$$

Now, a player 2 is a principal and a player 1 is an agent. A principal makes a take-or-leave-it offer.

- Hence, we will sometimes call $\gamma$ the bargaining power of a player 1.

# Wage Without Commitment

- Let me apply generalized Nash bargaining to our problem:

$$\max_{w} \left(W_i - U\right)^{\gamma} J_i^{1-\gamma}$$
$$s.t.\ W_i = \frac{w_i + sU}{\varrho + s},\ J = \frac{mp_l - w_i}{\varrho + s}$$

  where $i$ refers to $i$th pair.

- FOC:

$$\frac{\gamma \left(W_i - U\right)^{\gamma-1} J_i^{1-\gamma}}{\varrho + s} = \frac{(1-\gamma)\left(W_i - U\right)^{\gamma} J_i^{-\gamma}}{\varrho + s}$$
$$(1-\gamma)\left(W - U\right) = \gamma J$$

- The parameter $\gamma$ shows that labor share's of the total surplus that occupied job created:

$$W - U = \gamma \left[J + W - U\right] = \gamma S$$
$$J = (1-\gamma)\left[J + W - U\right] = (1-\gamma) S$$

# Wage Without Commitment

- $w$: Substituting the Definition of $S$ and $W$, we obtain the following equation.

$$
\begin{aligned}
W - U &= \gamma S \\
\frac{w + sU}{\varrho + s} - U &= \gamma \frac{mp_l - \varrho U}{\varrho + s} \\
w - \varrho U &= \gamma \left[ mp_l - \varrho U \right] \\
w &= \gamma \left[ mp_l - \varrho U \right] + \varrho U
\end{aligned}
$$

  - It shows that the wage is the reservation flow utility, $\varrho U$, plus the part of a flow surplus of this match, $\gamma \left[ mp_l - \varrho U \right]$:

# Wage Without Commitment

- $\varrho U$: Note that

$$\begin{aligned}\varrho U &= b + p(\theta)\theta[W - U] \\ &= b + p(\theta)\theta\gamma S\end{aligned}$$

Moreover, free entry condition shows the relationship between labor market tightness and the value of a firm:

$$\frac{\psi_s}{p(\theta)} + \psi_e = J = (1-\gamma) S \Rightarrow S = \frac{\frac{\psi_s}{p(\theta)} + \psi_e}{1-\gamma}$$

  - This relationship indicates that there is a relationship between the reservation flow utility of a worker, $\varrho U$, and labor market tightness, $\theta$:

# Wage Without Commitment

- Because of a search friction, there is the rent to share in this economy.
  1. If more jobs enter in the market, the reservation utility would be large because they can easily find new jobs and enjoy rent, $W - U$, in future (Positive trading externality).
  2. If an initial investment, $\frac{\psi_s}{p(\theta)} + \psi_e$, is large, it reduces the number of jobs' entries and increases the amount of rent obtained after the investment. Note that the average search cost $\frac{\psi_s}{p(\theta)}$ is influenced by labor market tightness $\theta$. Beucause a larger labour market tightness lowers a job's probability to find unemployed workers, $p(\theta)$, it increases the average search period, $\frac{1}{p(\theta)}$, and, therefore, the average search cost $\frac{\psi_s}{p(\theta)}$. Hence, a rise in $\theta$ increases the amount of future rent and, therefore, $\varrho U$.

# Wage Without Commitment

- This means

$$\varrho U = b + \frac{p(\theta)\,\theta\gamma\left[\frac{\psi_s}{p(\theta)} + \psi_e\right]}{1-\gamma}$$
$$= b + \frac{\gamma\left[\theta\psi_s + p(\theta)\,\theta\psi_e\right]}{1-\gamma}$$

# Wage Without Commitment

- The opportunity cost $\rho U$, is influenced not only by unemployment benefit $b$ but also by aggregate search costs per unemployed workers, $\theta\psi_s = \frac{vN\psi_s}{uN}$ and aggregate training cost per unemployed workers, $p(\theta)\theta\psi_e = \frac{mN\psi_e}{uN}$.
  - Because increases in search costs and training costs raise the amount of initial investment, they reduce the number of jobs' entries and increase the amount of rent obtained after the investment. Because unemployed workers expect that they can enjoy a part of this rent when they find their jobs, their reservation utility would be larger. In this way, search costs and training costs increase workers' reservation utility.

# Wage Without Commitment

- $w$: Substituting $\varrho U$ into the wage equation, we obtain

$$\begin{aligned} w &= \gamma\left[mp_l - \varrho U\right] + \varrho U \\ &= \gamma mp_l + (1-\gamma)\,\varrho U \\ &= \gamma mp_l + (1-\gamma)\left[b + \frac{\gamma\left[\theta\psi_s + p(\theta)\,\theta\psi_e\right]}{1-\gamma}\right] \\ &= \gamma R(\theta) + b,\ \ R(\theta) = mp_l - b + \theta\psi_s + p(\theta)\,\theta\psi_e \end{aligned}$$

  - wage, $w$, is influenced by marginal product of labor, $mp_l$, unemployment benefit $b$ plus aggregate search costs per unemployed workers, $\theta\psi_s$, aggregate training cost per unemployed workers, $p(\theta)\,\theta\psi_e$.
  - Note that $mp_l - b + \theta\psi_s + p(\theta)\,\theta\psi_e$ corresponds to the instantaneous surplus from employing one worker.

# Wage Without Commitment

- $\theta$: Using the derived wage, we can determine $\theta$, and therefore, $u$. Note that the free entry condition provides an information on the relationship between $\theta$ and $w$.

$$\begin{aligned} \frac{\psi_s}{p(\theta)} &= J - \psi_e \\ \frac{1}{p(\theta)} &= \frac{1}{\psi_s}\left[\frac{mp_l - w}{\varrho + s} - \psi_e\right] \end{aligned}$$

  - This equation shows that the high $w$ must be consistent with the low average duration to find workers, $\frac{1}{p(\theta)}$ and therefore, the low labor market tightness, $\theta$. When the wage, $w$, is high, the expected profit, $J$, is low. Therefore, a few vacancy, $v$, is posted. Hence, the labor market tightness, $\theta$, would be low.

# Wage Without Commitment

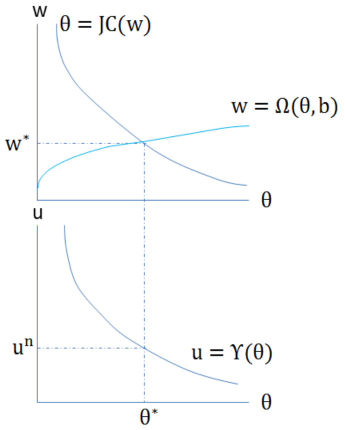
- Hence, the following two equations solves $w$ and $\theta$.

$$
\begin{aligned}
w &= \gamma R(\theta) + b \\
&\equiv \Omega(\theta, b), \ \Omega_1(\theta, b) > 0, \Omega_2(\theta, b) > 0 \\
\frac{1}{p(\theta)} &= \frac{1}{\psi_s}\left[\frac{mp_l - w}{\varrho + s} - \psi_e\right] \Rightarrow \theta = JC(w), JC'(w) < 0
\end{aligned}
$$

Assume the steady state. Then we know the natural rate of unemployment is determined by $\theta$:

$$
u^n = \frac{s}{p(\theta)\theta + s} \equiv \Upsilon(\theta), \ \Upsilon'(\theta) < 0
$$

## Natural Rate of Unemployment



# Wage Without Commitment

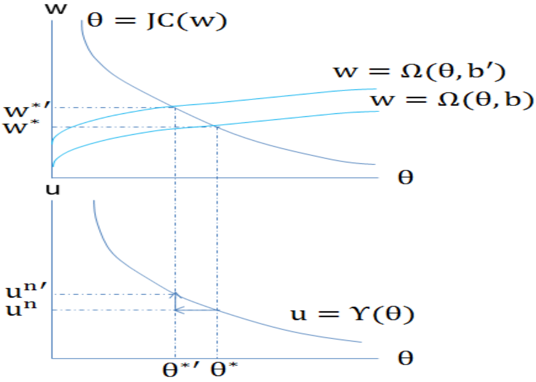
- $\frac{du}{db}$: an increase in $b$ pushes up an equilibrium wage, $w$, and therefore, reduces the expected profits and the number of vacancy. Hence, labor market tightness becomes lower $\theta$. Because the unemployment rate is stock variable, it does not change immediately. However, the lower labor market tightness implies that it is more difficult find a job. Following the dynamics of the unemployment rate,

$$\dot{u}_t = s\left[1 - u_t\right] - p\left(\theta\right)\theta u_t$$

the unemployment rate gradually goes up and reaches a new higher natural rate of unemployment.

# Wage Without Commitment

$$b \rightarrow b'(>b)$$



# Wage Without Commitment

- **Assignment:** Using the similar graph, discuss the economic logic of the following effects.
  1. Discuss the temporal and long run impacts of the productivity shock $mp_l$ on the wage and the unemployment rate.
  2. Discuss the temporal and long run impacts of search costs, $\psi_c$ and training costs, $\psi_e$, on the wage and the unemployment rate.

# Wage Without Commitment

- **Optimal unemployment rate:** If a part of unemployment is inevitable in a frictional economy, what is the optimal unemployment rate? Can market economy attain optimal unemployment rate?
- Because we have unemployed workers and employed workers in an economy, there are potentially many Pareto optimal allocations.
- We focus on those that maximize the sum of agent's utility, or equivalently, that maximize the present discounted value of output net of the disutility of work and search and training cost.

# Wage Without Commitment

- **Social Planner Problem:** Let us define a social planner model as follows.

$$\max_{\theta_t} \int_0^{\infty} \left\{ mp_l e_t + \left[ b - \left( \psi_s \theta_t + \psi_e p\left(\theta_t\right)\theta_t \right) \right] u_t \right\} e^{-\varrho t} dt$$

$$\begin{aligned} s.t.\ \dot{u}_t &= se_t - p\left(\theta_t\right)\theta_t u_t,\ \ u_0 \ is\ given \\ \dot{e}_t &= p\left(\theta_t\right)\theta_t u_t - se_t,\ \ e_0 = 1 - u_0 \ is\ given \end{aligned}$$

where

1. $mp_l e_t$ is the aggregate output per capita at date $t$.
2. $bu_t$ is the aggregate unemployment benefits (or the benefits from leisure) per capita at date $t$.
3. $\left( \psi_s \theta_t + \psi_e p\left(\theta_t\right)\theta_t \right) u_t$ is the sum of aggregate search cost and training cost per capita at date $t$.

# Wage Without Commitment

- Bellman equation

$$\varrho V\left(e_t, u_t\right) = \max_{\theta} \left\{ \begin{array}{c} mp_l e_t + \left[b - \left(\psi_s \theta_t + \psi_e p\left(\theta_t\right)\theta_t\right)\right] u_t \\ +V_u\left(e_t, u_t\right)\left[s e_t - p\left(\theta_t\right)\theta_t u_t\right] \\ +V_e\left(e_t, u_t\right)\left[p\left(\theta_t\right)\theta_t u_t - s e_t\right] \end{array} \right\}$$

## Lemma

$$\begin{aligned} V\left(e_t, u_t\right) &= \left(S^o + U^o\right) e_t + U^o u_t = S^o e_t + U^o \\ S^o &= \frac{mp_l - \varrho U^o}{\varrho + s} \\ \varrho U^o &= b + \max_{\theta} \left[ \begin{array}{c} p\left(\theta_t\right)\theta_t S^o \\ -\left(\psi_s \theta_t + \psi_e p\left(\theta_t\right)\theta_t\right) \end{array} \right] \end{aligned}$$

# Wage Without Commitment

- **Proof:** Guess $V(e_t, u_t) = (S^o + U^o)\, e_t + U^o u_t$

$$
\begin{aligned}
& \max_{\theta} \left\{ \begin{array}{c} mp_l e_t + \left[b - \left(\psi_s \theta_t + \psi_e p(\theta_t)\,\theta_t\right)\right] u_t \\ + U^o \left[s e_t - p(\theta_t)\,\theta_t u_t\right] \\ + (S^o + U^o) \left[p(\theta_t)\,\theta_t u_t - s e_t\right] \end{array} \right\} \\
= & \max_{\theta} \left\{ \begin{array}{c} \left[mp_l + s\left(U^o - (S^o + U^o)\right)\right] e_t + \\ \left[ \begin{array}{c} b + p(\theta_t)\,\theta_t \left[(S^o + U^o) - U^o\right] \\ - \left(\psi_s \theta_t + \psi_e p(\theta_t)\,\theta_t\right) \end{array} \right] u_t \end{array} \right\} \\
= & \begin{array}{c} \left[mp_l - s S^o\right] e_t + \\ \left\{ b + \max_{\theta} \left[ \begin{array}{c} p(\theta_t)\,\theta_t S^o \\ - \left(\psi_s \theta_t + \psi_e p(\theta_t)\,\theta_t\right) \end{array} \right] \right\} u_t \end{array}
\end{aligned}
$$

# Wage Without Commitment

- **Proof:** Hence,

$$\begin{aligned}
\varrho \left( S^o + U^o \right) &= mp_l - sS^o \Rightarrow S^o = \frac{mp_l - \varrho U^o}{\varrho + s} \\
\varrho U^o &= b + \max_{\theta} \left[ \begin{array}{c} p\left(\theta_t\right)\theta_t S^o \\ -\left(\psi_s \theta_t + \psi_e p\left(\theta_t\right)\theta_t\right) \end{array} \right]
\end{aligned}$$

# Wage Without Commitment

## Theorem

*The socially optimal labor market tightness $\theta^o$ must satisfy the following equation*

$$\begin{aligned}\frac{dp(\theta^o)\theta^o}{d\theta^o}\left[S^o - \psi_e\right] &= \psi_s \\ S^o &= \frac{R(\theta^o)}{\varrho + s + p(\theta^o)\theta^o},\end{aligned}$$

*where*

$$\begin{aligned}R(\theta^o) &= mp_l - b + \psi_s\theta^o + \psi_e p(\theta^o)\theta^o \\ \frac{dp(\theta^o)\theta^o}{d\theta^o} &= \left[1 + \frac{p'(\theta^o)\theta^o}{p(\theta^o)}\right]p(\theta^o)\end{aligned}$$

# Wage Without Commitment

- **Proof:**

$$\max_{\theta}\left\{p(\theta_t)\theta_t S^o - \left(\psi_s\theta_t + \psi_e p(\theta_t)\theta_t\right)\right\}$$

  - **FOC**

$$\begin{aligned}
\frac{dp(\theta^o)\theta^o}{d\theta^o}S^o &= \psi_s + \psi_e\frac{dp(\theta^o)\theta^o}{d\theta^o} \\
\left[\frac{p'(\theta^o)\theta^o + p(\theta^o)}{p(\theta^o)}\right]p(\theta^o)(S^o - \psi_e) &= \psi_s \\
\left[1 + \frac{p'(\theta^o)\theta^o}{p(\theta^o)}\right]p(\theta^o)(S^o - \psi_e) &= \psi_s
\end{aligned}$$

This equation means that $\theta^o$ is constant.

# Wage Without Commitment

- **Proof:** Previous lemma says that $S^o$ must satisfy,

$$
\begin{aligned}
S^o &= \frac{mp_l - \varrho U^o}{\varrho + s} \\
&= \frac{mp_l - \left\{ b + \left[ \begin{array}{c} p\left(\theta^o\right)\theta^o S^o \\ -\left(\psi_s \theta^o + \psi_e p\left(\theta^o\right)\theta^o\right) \end{array} \right] \right\}}{\varrho + s} \\
&= \frac{mp_l - b + \psi_s \theta^o + \psi_e p\left(\theta^o\right)\theta^o}{\varrho + s + p\left(\theta^o\right)\theta^o}
\end{aligned}
$$

- **Labor Market Tightness in the Search Equilibrium:** Let us analytically derive the labor market tightness at the market economy. Remember that the market equilibrium must satisfy two equations.

$$\begin{aligned}
w &= \gamma R(\theta) + b, \\
\frac{1}{p(\theta)} &= \frac{1}{\psi_s}\left[\frac{mp_l - w}{\varrho + s} - \psi_e\right] \\
R(\theta) &= mp_l - b + \theta\psi_s + p(\theta)\theta\psi_e
\end{aligned}$$

# Wage Without Commitment

## Theorem

*Labor Market Tightness in the search equilibrium, $\theta^*$, must satisfy the following equation.*

$$p(\theta^*)\left[(1-\gamma)S(\theta^*) - \psi_e\right] = \psi_s$$

$$S(\theta^*) = \frac{R(\theta^*)}{\rho + s + p(\theta^*)\theta^*}$$

$$R(\theta^*) = mp_l - b + \theta^*\psi_s + p(\theta^*)\theta^*\psi_e$$

# Wage Without Commitment

- **Proof:** Note that

$$\begin{aligned}\frac{1}{p(\theta)} &= \frac{1}{\psi_s}\left[\frac{mp_l - w}{\varrho + s} - \psi_e\right] \\ \frac{\psi_s}{p(\theta)} + \psi_e &= \frac{mp_l - w}{\varrho + s} \\ w &= mp_l - (\varrho + s)\left[\frac{\psi_s}{p(\theta)} + \psi_e\right]\end{aligned}$$

# Wage Without Commitment

- **Proof:** Hence $\theta$ must satisfy the following equation.

$$mp_l - (\varrho + s)\left[\frac{\psi_s}{p(\theta)} + \psi_e\right] = \gamma R(\theta) + b$$

It means that

$$\begin{aligned}
(1-\gamma) R(\theta) &= R(\theta) - \left\{ mp_l - b - (\varrho + s)\left[\frac{\psi_s}{p(\theta)} + \psi_e\right]\right\} \\
&= mp_l - b + \theta\psi_s + p(\theta)\theta\psi_e \\
&\quad - \left\{ mp_l - b - (\varrho + s)\left[\frac{\psi_s}{p(\theta)} + \psi_e\right]\right\} \\
&= \theta p(\theta)\left[\frac{\psi_s}{p(\theta)} + \psi_e\right] + (\varrho + s)\left[\frac{\psi_s}{p(\theta)} + \psi_e\right] \\
&= (\theta p(\theta) + \varrho + s)\left[\frac{\psi_s}{p(\theta)} + \psi_e\right]
\end{aligned}$$

# Wage Without Commitment

- **Proof:** Hence

$$
\begin{aligned}
(1-\gamma)\frac{R(\theta)}{p(\theta)\theta+\varrho+s} &= \frac{\psi_s}{p(\theta)}+\psi_e \\
(1-\gamma)\frac{R(\theta)}{p(\theta)\theta+\varrho+s}-\psi_e &= \frac{\psi_s}{p(\theta)} \\
(1-\gamma)\,p(\theta)\left[\frac{R(\theta)}{p(\theta)\theta+\varrho+s}-\psi_e\right] &= \psi_s
\end{aligned}
$$

# Wage Without Commitment

- **Comparison**
  - Social Optimal:

$$\left[1+\frac{p'(\theta^o)\theta^o}{p(\theta^o)}\right]p(\theta^o)\left[S^o-\psi_e\right]=\psi_s$$

$$S^o = \frac{R(\theta^o)}{\varrho+s+p(\theta^o)\theta^o}$$

  - Market Economy:

$$p(\theta^*)\left[(1-\gamma)S(\theta^*)-\psi_e\right]=\psi_s$$

$$S(\theta^*) = \frac{R(\theta^*)}{\rho+s+p(\theta^*)\theta^*}$$

where $R(\theta)=mp_l-b+\theta\psi_s+p(\theta)\theta\psi_e$

# Wage Without Commitment

**Theorem**

*Suppose that $\psi_e = 0$. Then the market economy attains social optimal if and only if the bargaining power of workers $\gamma$, equals to the elasticity of matching function with respect to $u$.*

$$\gamma = -\frac{p'(\theta)\,\theta}{p(\theta)} = \frac{m_u(u,v)\,u}{m(u,v)}.$$

*This is called Hosios Condition.*

# Wage Without Commitment

- **Proof:** Suppose $\psi_e = 0$. In this case
  - $\theta^o$ is chosen to satisfy
$$\left[1 + \frac{p'(\theta^o)\,\theta^o}{p(\theta^o)}\right] p(\theta^o)\, S^o = \psi_s$$
  - $\theta^*$ is chosen to satisfy
$$(1-\gamma)\, p(\theta^*)\, S(\theta^*) = \psi_s$$
  - Note that $S^o = S(\theta^o)$. Hence, the market economy attains social optimal if
$$\gamma = -\frac{p'(\theta)\,\theta}{p(\theta)}$$

- **Proof:** Note that
$$p(\theta) = m\left(\frac{1}{\theta}, 1\right).$$
Hence
$$\begin{aligned} p'(\theta) &= -m_u\left(\frac{1}{\theta}, 1\right)\frac{1}{\theta^2} \\ -\frac{p'(\theta)\theta}{p(\theta)} &= \frac{m_u\left(\frac{1}{\theta}, 1\right)\frac{1}{\theta^2}\theta}{\frac{m(u,v)}{v}} = \frac{m_u(u,v)\frac{v}{\theta}}{m(u,v)} = \frac{m_u(u,v)\,u}{m(u,v)} \end{aligned}$$
This is the elasticity of matching function with respect to $u$.

# Wage Without Commitment

- There is several reasons that a market economy is inefficient.
  - **Positive Trading Externality:** Posting vacancy makes it easy for unemployed workers to find new jobs. Because individual firms do not take into account this effect, the number of employed workers and, therefore, output is too small.
    - **Hold-up problem (we can discuss later):** Why do unemployed workers receive benefits when they can more easily find new jobs? When a firm find a workers, a worker obtains a part of rent from match. This is why unemployed workers are better off from high probability of finding a job. However, this means that a firm cannot obtain all the benefits from investment and, therefore, a firm hesitates to makes enough investment. Therefore, output is too small. This is called hold-up problem.
  - **Negative Trading externality**:Posting vacancy makes it difficult for other firms to find new workers (negative externality). Because individual firms do not take into account this effect, too many firms enter and society must incur too many search costs.

# Wage Without Commitment

- Because of positive or negative externality in the search model, labor market tightness in the market can be too large or too small. Therefore, the unemployment rate in the market equilibrium is also too many or too few. Hosios condition nicely balances these effects. Note that

$$\psi_s = \left[1 - \frac{m_u(u,v)\,u}{m(u,v)}\right] p(\theta^o)\,S^o = \frac{m_v(u,v)\,v}{m(u,v)} p(\theta^o)\,S^o$$

$$\psi_s = (1-\gamma)\,p(\theta^*)\,S(\theta^*) = \frac{J}{S} p(\theta^*)\,S(\theta^*)$$

This means that Hosios condition is rewritten by

$$\frac{J}{S} = 1 - \gamma = 1 - \frac{m_u(u,v)\,u}{m(u,v)} = \frac{m_v(u,v)\,v}{m(u,v)}.$$

This means that the private contribution of posting vacancy to a match is equal to social contribution to the mach.

# Wage Without Commitment

**Theorem**

*Suppose that* $\psi_s = 0$. *Then socially optimal unemployment rate is* $0$, *though a market economy maintains a positive unemployment rate. That is, the entry is too few under a market economy.*

# Wage Without Commitment

- **Proof:** Suppose that $\psi_s = 0$. Social optimal labor market tightness must satisfy

$$\begin{aligned} 0 &= \frac{dp(\theta^o)\theta^o}{d\theta^o}\left[S^o - \psi_e\right] \\ &= \frac{dp(\theta^o)\theta^o}{d\theta^o}\left[\frac{mp_l - b + p(\theta^o)\theta^o\psi_e}{\varrho + s + p(\theta^o)\theta^o} - \psi_e\right] \\ &= \frac{dp(\theta^o)\theta^o}{d\theta^o}\frac{mp_l - b - (\varrho + s)\psi_e}{\varrho + s + p(\theta^o)\theta^o}. \end{aligned}$$

As we implicitly assume $mp_l > b + \psi_e(\varrho + s)$ (otherwise nobody search), $\frac{dp(\theta^o)\theta^o}{d\theta^o} = 0$ or/and $p(\theta^o)\theta^o = \infty$. That is, $\theta = \infty$. Hence, the socially optimal unemployment rate is

$$\dot{u}_t = s[1 - u_t] - \lim_{\theta \to \infty} p(\theta_t)\theta_t u_t = -\infty$$

Hence, $u_t = 0$.

# Wage Without Commitment

- **Proof:** On the other hand, under a market economy, a labor market tightness, $\theta^*$ must satisfy

$$\begin{aligned}
0 &= p(\theta^*)\left[(1-\gamma)S(\theta^*)-\psi_e\right] \\
&= p(\theta^*)\left[(1-\gamma)\frac{mp_l-b+p(\theta^*)\theta^*\psi_e}{\rho+s+p(\theta^*)\theta^*}-\psi_e\right] \\
&= p(\theta^*)\left[\frac{(1-\gamma)(mp_l-b)-(\rho+s+\gamma p(\theta^*)\theta^*)\psi_e}{\rho+s+p(\theta^*)\theta^*}\right]
\end{aligned}$$

Because there exist $\theta^* < \infty$ that satisfy

$$(1-\gamma)(mp_l-b) = (\rho+s+\gamma p(\theta^*)\theta^*)\psi_e,$$

the unemployment eventually converges to the natural rate of unemployment:

$$u_t = \frac{s}{p(\theta^*)\theta^*+s} > 0.$$

# Wage Without Commitment

- Because there is no search cost, negative externality does not have any impacts on the society.
  - Although posting vacancy makes it difficult for other jobs to find new workers, without any search cost (=sunk cost a firm must incur before finding workers), there is no social loss from posting new jobs.
  - Therefore, the impact of positive externality (= hold-up problem) always dominates negative externality and the number of market economy in the market economy is too few. (Note that given the assumption of $mp_l > b + \psi_e (\rho + s)$, creating new job always increases net output. )

# Wage Without Commitment

## Theorem

*Suppose that $\psi_s = 0$. The market economy can attain social optimal and, therefore, there is no unemployment, if a worker does not have any bargaining power: $\gamma = 0$.*

## Proof.

*When $\gamma = 0$, from the proof of the previous theorem*

$$0 = p(\theta^*)\left[\frac{mp_l - b - (\rho + s)\psi_e}{\rho + s + p(\theta^*)\theta^*}\right]$$

*As we assume $mp_l > b + \psi_e(\rho + s)$ to insure positive entries, $p(\theta) = 0$. Therefore, $\theta = \infty$.* □

- If workers do not have any bargaining power, unemployed workers receive no benefits from the new match. Hence, there is no positive externality.

- Consider the following two period model: $t = 1$ or 2.
  - A supplier and a buyer try to trade a good which is suitable to a specific demand of a buyer. You can interpret a supplier as a worker, and a buyer as a firm. Both parties are risk neutral.
  - At the first period, a worker invests to improve human capital, $h$ with cost $C(h)$ where $C'(h) > 0$ and $C''(h) > 0$. Since human capital is useful only for this firm, if they fire the worker, the worker does not get anything.
  - At the second period, human capital yields output and the firm can sell the output at a price 1 and pays wage $w$.

# A Short Detour to Discussions on Hold Up Problem

- The key assumption is that two parties cannot make a contract at the first period. That is because
  1. both parties may not be able to foresee contingent future, or
  2. even though they can foresee, they may not be able to describe the contingent future, or
  3. even though they can describe, writing every contingency is quite costly.
- That is, both party can observe the level of investment at the first period, but it is not verifiable.

# A Short Detour to Discussions on Hold Up Problem

- **The first best:** The first best investment maximizes expected net benefit. If we assume that an agent does not discount future, then

$$\max_{I} \{h - C(h)\},$$

  - FOC

$$1 = C'\left(h^{best}\right).$$

- **Bargaining Problem:**
  - *The second period*: At the second period, everything becomes clear. A firm and a worker may be able to negotiate the wage $w$. Assume that if workers quite a job worker can get $U$ and if a firm fires the worker, a firm can get 0. Then the wage bargaining determines

$$w = U + \gamma (h - U)$$

    where $\gamma \in (0, 1)$ is the bargaining power of workers.
  - *The first period*: A worker knows that the wage will be determined as above. Given this knowledge his problem is

$$\begin{aligned} W &= \max_h [w - C(h)], \\ s.t.\ w &= U + \gamma (h - U) \end{aligned}$$

- FOC:

$$\gamma = C'(h^*)$$

Because $\gamma < 1$, and $C'' > 0$

$$h^* < h^{best}.$$

# A Short Detour to Discussions on Hold Up Problem

- That is, a worker does underinvestment. The reason is that since investment is specific to the firm, after making investment, the investment is sunk. If the firm fires the worker, investment is useless. Bargaining over wage reduces the marginal benefit from the investment. Since he knows it will happen, it discourages his investment. He will optimally reduce his investment. This is called "Hold-up problem."
- The above analysis suggest that an increase in $\gamma$ increases human capital accumulation. If $\gamma = 1$, $h^* = h^{best}$. Because a worker is a person to make investment decision, a larger bargaining power of a worker encourages human capital accumulation. Because $h^*$ is lower than social optimal value, larger human capital is welfare improving. So it is good to provide more power on workers.

# A Short Detour to Discussions on Hold Up Problem

- Let us illustrate the importance of an explicit contract at the first period.
- Suppose that a firm can offer an affine contract based on output: that is, $w = \alpha_0 + \alpha_1 h$.

$$\begin{aligned} V &= \max_{\alpha_0,\alpha_1} \left\{ h^{**} - \left( \alpha_0 + \alpha_1 h^{**} \right) \right\} \\ s.t.\ h^{**} &= \arg\max \left\{ \alpha_0 + \alpha_1 h - C\left(h\right) \right\} \quad (IC) \\ U &\leq \alpha_0 + \alpha_1 h^{**} - C\left(h^{**}\right) \quad (IR) \end{aligned}$$

# A Short Detour to Discussions on Hold Up Problem

- $h^{**}$

$$\alpha_1 = c'\left(h^{**}\right) \Rightarrow h^{**} = h^{**}\left(\alpha_1\right)$$

- Hence, a firm can reduce $\alpha_0$ without changing the incentive of workers until IR condition is bound.

$$\begin{aligned} U &= \alpha_0 + \alpha_1 h^{**}\left(\alpha_1\right) - C\left(h^{**}\left(\alpha_1\right)\right) \\ \alpha_0 + \alpha_1 h^{**}\left(\alpha_1\right) &= U + C\left(h^{**}\left(\alpha_1\right)\right) \end{aligned}$$

# A Short Detour to Discussions on Hold Up Problem

- Hence, the firm's problem can be rewritten as follows

$$V = \max_{\alpha_1, h^{**}} \left\{ h^{**}(\alpha_1) - C\left(h^{**}(\alpha_1)\right) - U \right\}$$

- FOC

$$\begin{aligned} 0 &= \left[1 - C'\left(h^{**}(\alpha_1)\right)\right] h^{**\prime}(\alpha_1) \\ 1 &= C'\left(h^{**}(\alpha_1)\right) \end{aligned}$$

- Note that $1 = C'\left(h^{best}\right)$. Hence,

$$h^{**}(\alpha_1) = h^{best}$$

- In this case,

$$\begin{aligned}\alpha_1 &= c'\left(h^{**}\left(\alpha_1\right)\right) = 1 \\ \alpha_0 &= U + C\left(h^{**}\left(\alpha_1\right)\right) - h^{**}\left(\alpha_1\right) \\ V &= -\alpha_0\end{aligned}$$

- That is, it is optimal for the firm to sell a job by price $-\alpha_0$ and let a worker receive all outcome from his own effort.

# Wage Commitment and Directed Search

- As shown in our previous discussion, when wage is determined by ex post bargaining, it is difficult to avoid hold-up problem, which is a version of positive externality.
- Although it is discussed that a firm can avoid hold-up problem if it can write an explicit contract on the investment, our current situation is more difficult because there is negative externality in the frictional economy.
- In fact, we know that if a firm can commit wage, the firm choose $w = b$ (Diamond Paradox). This is equivalent to the case, $\gamma = 0$. It causes too many entry.

# Wage Commitment and Directed Search

- The main reason for inefficiency is that search is undirected and, therefore, wage does not perform the role of allocating resources ex ante.
- If search is directed to a particular wage, when a firm posts its wage, the firm must consider that the wage influences the matching probability.
    - This may avoid Diamond paradox.
    - This can potentially internalize the effect of positive and negative externality.

# Wage Commitment and Directed Search

- Moen (1997) provides a convenient model of wage posting, which attains social optimal outcome.
- In his model, jobs post their wages before they enter the market. Unemployed workers direct their search to the most attractive jobs.
- Because high posted wage, $w$, attracts more applicants, which reduces workers' contact rate $p(\theta)\theta$ and raises the jobs' contact rate $p(\theta)$.
- Knowing this relationship between $w$ and $\theta$, unemployed workers decide which jobs for them to apply. Hence, all unemployed workers choose jobs that maximize their total sum of discounted utility, $U$.
- In other words, only jobs which post $w$ that maximizes $U$ survive in the equilibrium.

# Wage Commitment and Directed Search

- Each market is characterized by posted wage, $w$ and the labor market tightness $\theta(w)$ differs at each market. Define $\hat{W}$ as the set of market under which wages are posted by firms in an equilibrium. Given $\hat{W}$, unemployed workers choose which market they should join.
- Unemployed workers' optimal application given $\theta(w)$ and $\hat{W}$:

$$U = \max_{w \in \hat{W}} U(\theta(w), w)$$

where

$$\begin{aligned} \varrho U(\theta(w), w) &= b + p(\theta(w))\theta(w)[W(w) - U] \\ where\ \varrho W(w) &= w + s[U - W(w)]. \end{aligned}$$

# Wage Commitment and Directed Search

- $\varrho U(\theta(w), w)$
  1. $W(w) - U$

$$\begin{aligned} \varrho W(w) &= w + s[U - W(w)]. \\ \varrho[W(w) - U] &= w + s[U - W(w)] - \varrho U \\ W(w) - U &= \frac{w - \varrho U}{\varrho + s} \end{aligned}$$

  2. $\varrho U(\theta(w), w)$

$$\begin{aligned} \varrho U(\theta(w), w) &= b + p(\theta(w))\theta(w)[W(w) - U] \\ &= b + p(\theta(w))\theta(w)\frac{w - \varrho U}{\varrho + s} \end{aligned}$$

# Wage Commitment and Directed Search

- Market equilibrium condition: when all unemployed workers rationally choose a market, $\theta(w)$ must satisfy the following condition for any $w$.

$$U \geq U(\theta(w), w), \quad \frac{1}{\theta(w)} \geq 0, \quad \frac{[U - U(\theta(w), w)]}{\theta(w)} = 0.$$

  - This condition means that, for both $w \in \hat{W}$ and $w \notin \hat{W}$, the following $\theta(w)$ must be assigned. There are two cases.
    1. $w \leq \varrho U$: For all $\theta(w)$, $\varrho U(\theta(w), w) = b + p(\theta(w))\theta(w)\frac{w-\varrho U}{\varrho+s} \leq b$. Hence, nobody chooses the market $w$ as far as there is a market $w > b$, $\frac{1}{\theta(w)} = \frac{u}{v} = 0$.
    2. $w > \varrho U$: If $\frac{1}{\theta(w)} = \frac{u}{v} = 0$, then $\varrho U(\infty, w) = b + p(\infty)\infty\frac{w-\varrho U}{\varrho+s} = b + \infty > \varrho U$. Contradiction. Hence, $\frac{1}{\theta(w)} = \frac{u}{v} > 0$. In this case, $\theta(w)$ must satisfy $\varrho U = \varrho U(\theta(w), w) = b + p(\theta(w))\theta(w)\frac{w-\varrho U}{\varrho+s}$.

- Firms' profit maximization under free entries:

$$\begin{aligned} V &\leq 0, \ \textit{for all } w, \\ \hat{W} &= \{w : V = 0\} \end{aligned}$$

  - This free entry condition determines $\hat{W}$.

- Note that if $w \in \hat{W}$, $\theta(w)$ must satisfy

$$\begin{aligned} \psi_s &= p(\theta(w))[J(w) - \psi_e] \\ &= p(\theta(w))\left[\frac{mp_l - w}{\varrho + s} - \psi_e\right] \\ \frac{\psi_s}{p(\theta(w))} + \psi_e &= \frac{mp_l - w}{\varrho + s} \\ w &= mp_l - (\varrho + s)\left(\frac{\psi_s}{p(\theta(w))} + \psi_e\right) \end{aligned}$$

  - Note that $\psi_s > p(\infty)\left[\frac{mp_l - w}{\varrho + s} - \psi_e\right] = 0$. Hence, no firms offer wages $w \leq \varrho U$.

# Wage Commitment and Directed Search

- Summary of a directed search equilibrium:
  1. Profit 0 Condition, which determines $\hat{W}$ given $\theta(w)$.

$$\begin{aligned} w &= mp_l - (\varrho + s)\left(\frac{\psi_s}{p(\theta(w))} + \psi_e\right), \ w \in \hat{W} \\ w &> mp_l - (\varrho + s)\left(\frac{\psi_s}{p(\theta(w))} + \psi_e\right), \ if \ w \notin \hat{W} \end{aligned}$$

  2. Unemployment Workers' Optimal Application, which determines $U$ and $w$ given $\theta(w)$ and $\hat{W}$.

$$\varrho U = \max_{w \in \hat{W}} \left\{ b + p(\theta(w))\,\theta(w)\,\frac{w - \varrho U}{\varrho + s} \right\}$$

  3. Market Equilibrium Condition, which determine $\theta(w)$.

$$\begin{aligned} \varrho U &= b + p(\theta(w))\,\theta(w)\,\frac{w - \varrho U}{\varrho + s}. \ if \ w > \varrho U \\ \theta(w) &= \infty \ if \ w \leq \varrho U \end{aligned}$$

# Wage Commitment and Directed Search

- Equilibrium conditions imply that when $w \in \hat{W}$, it must satisfy free entry condition and must maximize the utility of unemployed workers. Hence, the problem can be shown to be equivalent to

$$
\begin{aligned}
\varrho U &= \max_{w,\theta} \varrho U(\theta, w) \\
s.t. \quad w &= mp_l - (\varrho + s)\left(\frac{\psi_s}{p(\theta)} + \psi_e\right) \\
\text{where } \varrho U(\theta, w) &= b + p(\theta)\,\theta\,\frac{w - \varrho U}{\varrho + s}
\end{aligned}
$$

- Eliminating $w$,

$$
\begin{aligned}
& \max_{\theta}\left\{b+p(\theta)\theta\frac{\left[mp_l-(\varrho+s)\left(\frac{\psi_s}{p(\theta)}+\psi_e\right)\right]-\varrho U}{\varrho+s}\right\} \\
= & \max_{\theta}\left\{b+p(\theta)\theta\left[\frac{mp_l-\varrho U}{\varrho+s}-\left(\frac{\psi_s}{p(\theta)}+\psi_e\right)\right]\right\} \\
= & \; b+\max_{\theta}\left\{p(\theta)\theta\frac{mp_l-\varrho U}{\varrho+s}-\left(\theta\psi_s+p(\theta)\theta\psi_e\right)\right\}
\end{aligned}
$$

# Wage Commitment and Directed Search

- Therefore, the original problem can be rewritten as follows.

$$\begin{aligned} \varrho U &= b + \max_{\theta} \left[ \begin{array}{c} p(\theta)\theta S \\ -\left(\psi_s \theta + \psi_e p(\theta)\theta\right) \end{array} \right] \\ S &= \frac{mp_l - \varrho U}{\varrho + s} \end{aligned}$$

- Remember that optimal unemployment can be found as a solution of

$$\begin{aligned} V(e_t, u_t) &= (S^o + U^o) e_t + U^o u_t = S^o e_t + U^o \\ S^o &= \frac{mp_l - \varrho U^0}{\varrho + s} \\ \varrho U^0 &= b + \max_{\theta} \left[ \begin{array}{c} p(\theta_t)\theta_t S^o \\ -\left(\psi_s \theta_t + \psi_e p(\theta_t)\theta_t\right) \end{array} \right] \end{aligned}$$

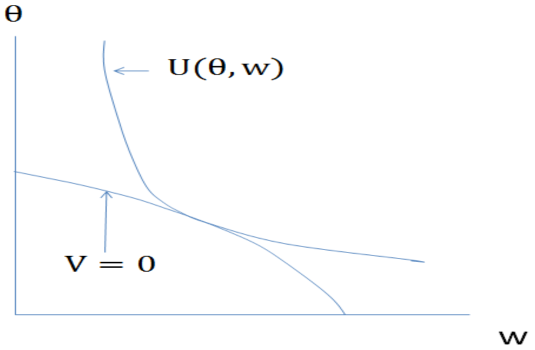
# Wage Commitment and Directed Search

## Theorem

*The competitive search equilibrium attains an social optimal allocation.*

- Note that we have two externality: positive externality (= hold up problem ) and negative externality.
- Because of profit 0 condition, workers obtain all benefits and costs of externality. But, when unemployed workers apply particular jobs, they take into account not only wage payment, $w$, but also the expected time to wait before finding the job, $\frac{1}{p(\theta)\theta}$. In this way, the externality is internalized.

Directed Search



# Final Words

- Congratulation! You finish all subjects in this lecture.
- Topics in macroeconomics are broad. So there are many subjects this lecture did not cover such as monetary economy, international macroeconomics and endogenous growth, heterogeneity among agents, incomplete market and so on.
- However, I guarantee that you have already studied the most important fundamental methods to analyze these problems. So you will be able to read many textbooks and papers in macroeconomics by yourself. Try it.