Answer for Homework 2: Modern Macroeconomics I*

1. Consider a constant returns to scale production function $Y = F(K, L)$.

   (a) What is the definition of the constant returns to scale?

   Answer Production function is constant return to scale if

   $$tF(K,L) = F(tK,tL), \quad \text{for any } t > 0.$$

   (b) Show that a firm cannot earn economic profits.

   Answer The firm's profit maximization problem is given by

   $$\max_{K,L} \{PF(K,L) - RK - WL\}.$$

   Since the production function F exhibits constant returns to scale, we can write the firm's profit as follows:

   $$\begin{aligned}
   &PF(K,L) - RK - WL \\
   =&PLF(\frac{K}{L}, \frac{L}{L}) - RK - WL \\
   =&PLF(k,1) - RK - WL \\
   =&PL \cdot \left[F(k,1) - \frac{R}{P}\frac{K}{L} - \frac{W}{P}\frac{L}{L}\right] \\
   =&PL \cdot [f(k) - rk - w]
   \end{aligned}$$

   where $k \equiv \frac{K}{L}$ is capital per worker, $f(k) \equiv F(k,1)$ is production technology per worker, $r \equiv \frac{R}{P}$ is real rental price, and $w \equiv \frac{W}{P}$ is real wage rate[1].
   Hence the firm's problem is written as follows:

   $$\max_{k,L} \{PL \cdot [f(k) - rk - w]\}.$$

   First, we maximize the profits with respect to capital per worker k,

   $$\max_{k} \{f(k) - rk - w\}.$$

*I thank to Hiroshi Kitamura and Wataru Tamura who made these sample answers.

[1] Note that "$k \equiv K/L$" denotes "we define k as K/L".

From the first-order condition, the optimal level of k is given by

$$f'(k^*) = r.$$

Next, we fix capital per worker $k = k^*$ and maximize the profit with respect to labor input L,

$$\max_L \{PL \cdot (f(k^*) - rk^* - w)\}.$$

Now consider the following three cases;

i. If $f(k^*) - rk^* > w$, the firm is better off increasing its labor demand up to infinity, $L \to \infty$. In this case, the labor demand must exceed the labor supply.

ii. If $f(k^*) - rk^* < w$, the firm employs no worker $L = 0$. In this case, the labor supply must exceed the labor demand.

iii. If $f(k^*) - rk^* = w$, the firm cannot make profits since $\Pi = PL \cdot 0$.

In order to clear the labor market (demand equals supply), the real rental price and the real wage (r, w) must satisfy $f(k^*) - rk^* = w$, which implies that the firm earns economic profits in equilibrium.

(c) In reality, we observe profits in a market. How can we reconcile the theory with this evidence?

Answer We observe profits in market because the concept of economic profit differs from usual accounting profits. In reality, a firm's owner owns capital. Therefore,

$$\text{Accounting profits} = \pi + rK$$

where π is economic profit. When production function is the constant return to scale, $\pi = 0$ and accounting profits becomes $rK > 0$. Hence, observable accounting profits is approximated by the return to capital.

2. Show whether or not the following production function are constant return to scale in K and L?

(a)

$$Y = K^\alpha L^\beta, 0 < \alpha + \beta < 1$$

Answer For any $\gamma > 0$,

$$\begin{aligned} (\gamma K)^\alpha (\gamma L)^\beta &= \gamma^{\alpha+\beta} K^\alpha L^\beta \\ &= \gamma^{\alpha+\beta} Y \\ &\neq \gamma Y \end{aligned}$$

Hence, this is not constant return to scale in K and L.

(b)

$$Y = [F(K, L)]^\alpha, \ 0 < \alpha < 1,$$

where $F(K, L)$ is constant return to scale in K and L.

Answer Since F is constant return to scale in K and L, for any $\gamma > 0$

$$\begin{aligned} [F(\gamma K, \gamma L)]^{\alpha} &= [\gamma F(K, L)]^{\alpha} \\ &= \gamma^{\alpha} [F(K, L)]^{\alpha} \\ &\neq \gamma Y \end{aligned}$$

Hence, this is not constant return to scale in K and L.

(c)

$$Y = [(1-\theta) K^{\rho} + \theta L^{\rho}]^{\frac{1}{\rho}}, \ 0 < \theta < 1$$

Answer For any $\gamma > 0$

$$\begin{aligned} [(1-\theta)(\gamma K)^{\rho} + \theta (\gamma L)^{\rho}]^{\frac{1}{\rho}} &= [(1-\theta)\gamma^{\rho} K^{\rho} + \theta \gamma^{\rho} L^{\rho}]^{\frac{1}{\rho}} \\ &= [((1-\theta) K^{\rho} + \theta L^{\rho}) \gamma^{\rho}]^{\frac{1}{\rho}} \\ &= \gamma [(1-\theta) K^{\rho} + \theta L^{\rho}]^{\frac{1}{\rho}} \\ &= \gamma Y \end{aligned}$$

Hence, this is constant return to scale in K and L.

(d)

$$Y = F(K, L) + G(K, L)$$

where $F(K, L)$ and $G(K, L)$ are constant return to scale in K and L.

Answer Since $F(K, L)$ and $G(K, L)$ are constant return to scale in K and L, for any $\gamma > 0$,

$$\begin{aligned} F(\gamma K, \gamma L) + G(\gamma K, \gamma L) &= \gamma F(K, L) + \gamma G(K, L) \\ &= \gamma [F(K, L) + G(K, L)] \\ &= \gamma Y \end{aligned}$$

Hence, this is constant return to scale in K and L.

(e)

$$Y = [F(K, L)]^{\alpha} L^{(1-\alpha)}, \ 0 < \alpha < 1$$

where $F(K, L)$ is constant return to scale in K and L.

Answer Since $F(K, L)$ is constant return to scale, for any $\gamma > 0$

$$\begin{aligned} [F(\gamma K, \gamma L)]^{\alpha} (\gamma L)^{(1-\alpha)} &= [\gamma F(K, L)]^{\alpha} (\gamma L)^{(1-\alpha)} \\ &= \gamma [F(K, L)]^{\alpha} L^{(1-\alpha)} \\ &= \gamma Y \end{aligned}$$

Hence, this is constant return to scale in K and L.

3. Answer the following questions.

(a) When a firm maximizes its profits by choosing the amount of labor, the marginal product of labor is equal to the real wage rate. Explain its economic (intuitive) reason behind this mathematical result.

Answer The marginal product of labor is written as

$$\text{MPL} \simeq \frac{F(K, L+\Delta) - F(K, L)}{\Delta}$$

where $\Delta > 0$ is a very small positive number. In words, if the firm increases the labor input from L to $L + \Delta$, its output increases by $\text{MPL} \cdot \Delta$.
Now we show that if $\text{MPL} \neq w$, a firm has an incentive to change its production plan, that implies such a situation is not an equilibrium.

- Consider the case in which the marginal product of labor exceeds the real wage rate ($\text{MPL} > w$). Then the firm is better off increasing the labor input by Δ. To see this, suppose that $\text{MLP} > w$. Then

$$\begin{aligned}
&\Leftrightarrow \text{MPL} \cdot \Delta > w \cdot \Delta \\
&\Leftrightarrow F(K, L+\Delta) - F(K, L) > w \cdot \Delta \\
&\Leftrightarrow F(K, L+\Delta) - w\Delta > F(K, L) \\
&\Leftrightarrow F(K, L+\Delta) - rK - w(L+\Delta) > F(K, L) - rK - wL.
\end{aligned}$$

  The left-hand side of the last inequality is the profits when the firm chooses $(L+\Delta)$ and the right-hand side of the last inequality is the profits when the firm chooses L. Hence the firm has an incentive to employ more workers.

- Similarly, if the marginal product of labor is less than the real wage rate ($\text{MPL} < w$), the firm has an incentives to employ less workers.

Therefore, in equilibrium, the marginal product of labor must equal the real wage rate.

(b) When a labor market is competitive, the demand for labor is equal to the supply of labor in a market equilibrium. Explain a rationale for this definition of the equilibrium.

Answer When there exists an excess labor demand ($D_L(w) > S_L(w)$), then firms rise wage, w, to employ more workers. On the other hand, when there exists an excess labor supply ($D_L(w) < S_L(w)$), then workers accept a decrease in wage to be employed. Therefore, the excess labor demand leads to an increase in wage but the excess labor supply leads to a decrease in wage. It is straightforward to see that the equilibrium in which the demand for labor is equal to the supply of labor can be obtained by the above process.

(c) When a financial market is competitive (and any risk can be diversified), then the returns on investment are expected to be the same across investment opportunities in a market equilibrium. Explain a rationale for the definition of the equilibrium.

Answer If there is a higher return on investment than the others, firms have an incentive to invest in this investment opportunity instead of the others. When the return is a decreasing function of investment, an increase in the investment leads to a decrease in the return. Therefore, the return on this investment becomes smaller and the returns on the others become higher. It is easy to see that the returns on investment are the same across investment opportunities in the market equilibrium.

(d) In our lecture note, the price of goods produced is assumed to be 1. Explain a rationale for this assumption.

Answer Since our concern here is the relative price of goods, this assumption does not lose the generality.

4. An economy described by the neoclassical growth model has the following production function:

$$Y = K^{\alpha} (TL)^{1-\alpha}, \; 0 < \alpha < 1$$

where Y is GDP, K is capital stock, T is productivity and L is the number of labor.

(a) Assume that a firm maximizes its profit given wage, w and the rental cost of capital, r. Show that

$$\alpha = \frac{rK}{Y}, \quad 1 - \alpha = \frac{wL}{Y}$$

Answer The firm's profit maximization problem is as follows:

$$\begin{aligned}
\max_{K,L} &= \left\{ PK^{\alpha}(TL)^{1-\alpha} - RK - WL \right\} \\
&= P(TL) \left\{ \left(\frac{K}{TL} \right)^{\alpha} - \frac{R}{P}\frac{K}{TL} - \frac{W}{P}\frac{L}{TL} \right\} \\
&= P(TL) \left\{ k_e^{\alpha} - rk_e - w/T \right\}.
\end{aligned}$$

where k_e denotes the capital stock per unit of effective labor.
First, we consider the maximization with respect to k_e, and then consider the maximization with respect to L.

i. First solve

$$\max_{k_e} \left\{ k_e^{\alpha} - rk_e - w/T \right\}.$$

From the first-order condition, we obtain

$$\begin{aligned}
&\alpha k_e^{\alpha-1} - r = 0 \\
\Leftrightarrow & rk_e = \alpha k_e^{\alpha} \\
\Leftrightarrow & r\frac{K}{TL} = \alpha \left(\frac{K}{TL} \right)^{\alpha} \\
\Leftrightarrow & rK = \alpha K^{\alpha}(TL)^{1-\alpha} \\
\Leftrightarrow & rK = \alpha Y.
\end{aligned}$$

ii. Next we consider the maximization problem with respect to L. However, in equilibrium, the firm cannot make profits. Recall that in equilibrium the real rental price and the real wage rate (r, w) must satisfy

$$(k_e^*)^\alpha - rk_e^* - w/T = 0.$$

From this equation, we can obtain $w = (1-\alpha)Y/L$. The derivation is as follows:

$$\begin{aligned}
&(k_e^*)^\alpha - rk_e^* - w/T = 0 \\
\Leftrightarrow &\left(\frac{K}{TL}\right)^\alpha - r\frac{K}{TL} = w/T \\
\Leftrightarrow &\left(\frac{K}{TL}\right)^\alpha (TL) - rK = wL \\
\Leftrightarrow &Y - rK = wL \\
\Leftrightarrow &(1-\alpha)Y = wL
\end{aligned}$$

where the last equality follows from $rK = \alpha Y$ in equilibrium.

Alternative derivation

Here I explain an alternative derivation using partial derivatives. Define $\Pi(K, L) \equiv PK^\alpha(TL)^{1-\alpha} - RK - WL$. Then the profit maximization problem is written as

$$\max_{K,L} \; = \Pi(K, L).$$

First order conditions are given by

$$\frac{\partial \Pi(K, L)}{\partial K} = 0 \quad \text{and} \quad \frac{\partial \Pi(K, L)}{\partial L} = 0.$$

$$\begin{aligned}
&\frac{\partial \Pi(K, L)}{\partial K} = P \cdot \alpha K^{\alpha-1}(TL)^{1-\alpha} - R = 0 \\
\Leftrightarrow &P \cdot \alpha K^\alpha K^{-1}(TL)^{1-\alpha} = R \\
\Leftrightarrow &\alpha Y = rK.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\frac{\partial \Pi(K, L)}{\partial L} = P \cdot K^\alpha T^{1-\alpha}(1-\alpha)L^{-\alpha} - W = 0 \\
\Leftrightarrow &P \cdot (1-\alpha)K^\alpha(TL)^{1-\alpha}L^{-1} = W \\
\Leftrightarrow &P \cdot (1-\alpha)Y = WL \\
\Leftrightarrow &(1-\alpha)Y = wL.
\end{aligned}$$

Solving these equation with respect to α and $1-\alpha$, we have

$$\alpha = \frac{rK}{Y}, \quad 1-\alpha = \frac{wL}{Y}.$$