Answer for Homework 3: Modern Macroeconomics I*

1. If two countries have the same saving rate, the same population growth and the same technology, then a Solow model predicts that both countries converge to the same steady state and attain the same GDP per capita in the long run. Explain economic (intuitive) reason behind this prediction.

   Answer Suppose that two countries have the same saving rate, the same population growth, and the same technology and differ only in the initial economic conditions such as initial capital stock and initial population. Suppose country A has a less capital stock per worker than country B (that is, $k_A < k_B$). Since the marginal productivity of capital is decreasing in capital stock per worker, country A has a higher marginal productivity of capital than country B (that is, $\text{MPK}_A = f'(k_A) > f(k_B) = \text{MPK}_B$). In this case, the growth rate of capital stock per worker in country A is higher than the one in country B. Intuitively, the less developed country accumulate capital stock faster than the developed one and these growth rates eventually converge to the rate of population growth.

2. Answer the following questions.

   (a) Kaldor found 6 robust facts for the long run behavior of aggregate data. What are the Kaldor's Stylized facts?

   Answer Kaldor's stylized facts are as follows:

   1 The growth rate of GDP per capita is nearly constant.

   2 The growth rate of capital per capita is nearly constant.

   3 The growth rate of output per worker differs substantially across countries.

   4 The rate of return to capital is nearly constant.

   5 The ratio of physical capital to output is nearly constant.

   6 The shares of labor and physical capital are nearly constant.

   (b) Show how the neoclassical growth model in the steady state explains the Kaldor's Stylized Facts.

   Answer (Revised) Note that on the steady state, k_e^* is constant and therefore $y_e^* = f(k_e^*)$ and $f'(k_e^*)$ are constant

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1 Note that

$$g_y = g_{y_e^* T} \approx g_{y_e^*} + g_T = g_T = g.$$

Since g is constant, the growth rate of GDP per capita is constant in the neoclassical growth model.
Alternative
GDP per capita is written as $y_t = \frac{F(K_t, T_t L_t)}{L_t} = T_t f(k_{et})$. Since the capital stock per effective worker is constant in the steady state ($k_{et} = k_e^*$), the growth rate of y_t is given by

$$\begin{aligned} g_y = \frac{y_{t+1} - y_t}{y_t} &= \frac{T_{t+1} f(k_e^*) - T_t f(k_e^*)}{T_t f(k_e^*)} \\ &= \frac{T_{t+1} - T_t}{T_t} = g. \end{aligned}$$

Hence we observe that the growth rate of GDP per capita is constant in the model.

2 Note that

$$g_k = g_{k_e^* T} \approx g_{k_e^*} + g_T = g_T = g.$$

Since g is constant, the growth rate of capital per capita is constant in the neoclassical growth model.
Alternative
Capital stock per capita is written as $k_t = \frac{K}{L} = T_t k_{et}$. Since the capital stock per effective worker is constant in the steady state ($k_{et} = k_e^*$), the growth rate of k_t is given by

$$\begin{aligned} g_k = \frac{k_{t+1} - k_t}{k_t} &= \frac{T_{t+1} k_e^* - T_t k_e^*}{T_t k_e^*} \\ &= \frac{T_{t+1} - T_t}{T_t} = g. \end{aligned}$$

Hence we observe that the growth rate of capital per capita is constant in the model.

3 We have observed that, in steady state, the growth rate of output per capita coincides with the growth rate of technology. Hence the model explains the difference in the growth rate of output per capita by the difference in the exogenous growth rate of technology.

4 From the firm's profit maximization problem[1], the capital stock per effective

[1] Recall that the profit maximization problem is

$$\begin{aligned} &\max_{K,L} \ \{PF(K, TL) - RK - WL\} \\ \Rightarrow\ &\max_{k_e, L} \ PTL \cdot \{f(k_e) - rk_e - w/T\}. \end{aligned}$$

worker and the real rental price should satisfies

$$f'(k_{et}) = r_t.$$

In the steady state, the capital per effective worker is constant ($k_{et} = k_e^*$), the rate of return to capital (real rental price) is constant in the neoclassical growth model.

5 Note that

$$\frac{K_t}{Y_t} = \frac{k_e^* T_t N_t}{y_e^* T_t N_t} = \frac{k_e^*}{y_e^*}$$

Since both k_e^* and y_e^* are constant in the steady state, the ratio of physical capital to output is constant in the neoclassical growth model.

6 Note that $r_t = f(k_e^*)$ in steady state. Then,

$$\frac{r_t K_t}{Y_t} = f'(k_e^*) \frac{k_{et}}{y_t} = f'(k_e^*) k_e^* / y_e^* = \text{constant}$$

Since

$$1 = \frac{r_t K_t + w_t L_t}{Y_t},$$

we have

$$\frac{w_t L_t}{Y_t} = 1 - \frac{r_t K_t}{Y_t} = \text{constant}$$

(c) Describe 5 development facts stated by Parente and Prescott(1993) and Durlauf and Quah(1998).

Answer 5 development facts stated by Parente and Prescott(1993) and Durlauf and Quah(1998) are as follows:

1 Income difference across countries is large.

2 Wealth distribution has shifted up.

3 Relative Income distribution does not show convergence.

4 There have been development miracles and disasters.

5 Relative Income distribution across countries shows two peaks.

3. Consider the following production function

$$Y_t = A_t K_t^{\alpha} (N_t)^{1-\alpha}$$

where Y_t is GDP, K_t is capital stock, N_t is the number of population and A_t is Solow residual at date t. Assume that a firm maximizes its profit and every market is competitive. Assume that we can obtain data about Y_t, K_t, N_t. I also assume that it is possible to obtain data about the rental price of capital and the wage rate.

(a) Show how g_A can be estimated?

Answer Note that

$$g_Y = g_A + \alpha g_K + (1-\alpha) g_N$$

Since a firm maximizes its profits in the competitive market,

$$\begin{aligned}\alpha &= \frac{r_t K_t}{Y_t} \\ 1-\alpha &= \frac{w_t N_t}{Y_t}\end{aligned}$$

Hence,

$$g_A = g_Y - \frac{r_t K_t}{Y_t} g_K - \frac{w_t N_t}{Y_t} g_N$$

(b) Suppose that it is difficult to estimate the rental price of capital. Can you still estimate g_A? Show your estimation strategy.

Answer Yes, we can estimate g_A even when it is difficult to estimate r_t. Since the production function is constant return to scale, we have

$$Y_t = r_t K_t + w_t L_t$$

Then we have

$$\frac{r_t K_t}{Y_t} = 1 - \frac{w_t L_t}{Y_t}$$

Hence, we have

$$\begin{aligned}g_A &= g_Y - \left(1 - \frac{w_t L_t}{Y_t}\right) g_K - \frac{w_t N_t}{Y_t} g_N \\ &= g_Y - g_K + \frac{w_t L_t}{Y_t} \left(g_K - g_N\right)\end{aligned}$$

4. Suppose that Japan and the US has the same production function

$$Y_t = K_t^{\alpha} \left(T_t N_t\right)^{(1-\alpha)}$$

where Y_t is output, K_t is capital stock, T_t is the labor augmenting technology and N_t is the number of workers. Assume that the growth rate of technology, g, and the depreciation rate, δ, and the growth rate of the number of workers, n, in Japan are the same as those in the US. Assume that $\alpha = \frac{1}{3}$.

(a) Derive the steady state value of GDP per workers as a function of the saving rate, s, the growth rate of technology, g, the depreciation rate, δ and the the growth rate of the number of workers, n, and the current level of technology, T_t.

Answer Refer to lecture slide. You can derive the following dynamics of capital per unit of effective labor.

$$k_{e,t+1} = \frac{s y_{et} + (1-\delta) k_{et}}{(1+g)(1+n)} \tag{1}$$

where $y_{et} = \frac{Y_t}{T_t N_t}$ and $k_{et} = \frac{K_t}{T_t N_t}$. It is easy to see that $y_{et} = k_{et}^{\alpha}$ and equation

(1) becomes

$$k_{e,t+1} = \frac{sk_{et}^{\alpha} + (1-\delta)\,k_{et}}{(1+g)\,(1+n)}.$$

In the steady state, $k_{et+1} = k_{et} = k_e^*$ and we have

$$k_e^* = \frac{s\,(k_{et}^*)^{\alpha} + (1-\delta)\,k_e^*}{(1+g)\,(1+n)}.$$

By solving this equation with respect to k_e^*

$$k_e^* = \left[\frac{s}{g+n+\delta+gn}\right]^{\frac{1}{1-\alpha}}.$$

Let $\frac{Y_t^*}{N_t^*}$ be the steady state value of GDP per workers. Since $y_{et} = \frac{Y_t}{T_t N_t}$ and

$$\begin{aligned}\frac{Y_t^*}{N_t^*} &= (k_e^*)^{\alpha}\,T_t \\ &= \left[\frac{s}{g+n+\delta+gn}\right]^{\frac{\alpha}{1-\alpha}} T_t.\end{aligned}$$

Since $\alpha = \frac{1}{3}$, we have

$$\frac{Y_t^*}{N_t^*} = \left[\frac{s}{g+n+\delta+gn}\right]^{\frac{1}{2}} T_t$$

(b) Suppose that Japan has 2 times larger saving rate than US and they have the same current level of technology. How much does income per capita differ on the steady state?

Answer Since Japan has 2 times higher saving rate,

$$\begin{aligned}\frac{\frac{Y_t^*}{N_t^*}JP}{\frac{Y_t^*}{N_t^*}US} &= \frac{\left[\frac{s_{JP}}{g+n+\delta+gn}\right]^{\frac{1}{2}} T_t}{\left[\frac{s_{US}}{g+n+\delta+gn}\right]^{\frac{1}{2}} T_t} \\ &= \left[\frac{s_{JP}}{s_{US}}\right]^{\frac{1}{2}} \\ &= 2^{\frac{1}{2}}\end{aligned}$$

Therefore, Japan has $2^{\frac{1}{2}}$ times higher GDP per capita.

(c) Assume that Japan and the US have the same saving rate, but the current level of technology in the US is 1.5 times larger than that in Japan. How much does income per capita differ on the steady state?

Answer Since US has 2 times larger productivity than Japan,

$$\frac{\frac{Y_t^*}{N_t^*}_{US}}{\frac{Y_t^*}{N_t^*}_{JP}} = \frac{\left[\frac{s}{g+n+\delta+gn}\right]^{\frac{1}{2}} T_t^{US}}{\left[\frac{s}{g+n+\delta+gn}\right]^{\frac{1}{2}} T_t^{JP}} = \frac{T_t^{US}}{T_t^{JP}} = 1.5$$

Therefore, the US has 1.5 times larger GDP per capita than Japan.

(d) A developed country has a saving rate of 30 percent and a population growth rate of 2 percent per year. Suppose that a developing country has a saving rate of 10 percent and a population growth rate of 5 percent. Suppose that initial technology of the developed country is 10 times higher than that of the developing country and that both country has the same productivity growth and depreciation rate: $g = 0.02$ and $\delta = 0.03$. Assume that $\alpha = \frac{1}{3}$. How much is GDP per capita in the developed country larger than that in the developing country in the steady state?

Answer (Revised) GDP per capita in developed country is

$$\begin{aligned}\frac{Y_t^*}{N_t^*}_{Developed} &= \left[\frac{s}{g+n+\delta+gn}\right]^{\frac{\alpha}{1-\alpha}} T_t^{Developed} \\ &= \left[\frac{0.3}{0.02+0.02+0.03+0.02\cdot 0.02}\right]^{\frac{1}{2}} (1+g)^t T_0^{developed} \\ &= \left[\frac{3000}{704}\right]^{\frac{1}{2}} (1+g)^t T_0^{Developed}\end{aligned}$$

On the other hand, GDP per capita in developing country is

$$\begin{aligned}\frac{Y_t^*}{N_t^*}_{Developing} &= \left[\frac{s}{g+n+\delta+gn}\right]^{\frac{\alpha}{1-\alpha}} T_t^{Developing} \\ &= \left[\frac{0.1}{0.02+0.05+0.03+0.02\cdot 0.05}\right]^{\frac{1}{2}} (1+g)^t T_0^{Developing} \\ &= \left[\frac{100}{101}\right]^{\frac{1}{2}} (1+g)^t T_0^{Developing}\end{aligned}$$

$$\frac{\frac{Y_t^*}{N_t^*}_{Developed}}{\frac{Y_t^*}{N_t^*}_{Developing}} = \frac{5}{4}\sqrt{\frac{3030}{11}} \approx 20.75$$

Therefore, GDP per capita in the developed country is 20.75 times larger than that in the developing country in the steady state.

(e) Derive a regression equation that shows $\log \frac{Y}{L}$ as a functions of $\log s$ and

$\log(n+g+\delta+ng)$.

Answer In the equilibrium, $L_t = N_t$. Then,

$$\begin{aligned}\log\frac{Y_t}{L_t} &= \log\left(\left(\frac{s}{g+n+\delta+gn}\right)^{\frac{1}{2}} T_t\right) \\ &= \log T_t + \frac{1}{2}\log s - \frac{1}{2}\log(n+g+\delta+ng).\end{aligned}$$