Modern Macroeconomics II

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1 Introduction

We have studied that the neoclassical growth model is summarized by the following 3 equations.

$$\begin{aligned} \dot{K}_t &= sF(K_t, T_t L_t) - \delta K_t \\ \dot{T}_t &= gT_t \\ \dot{L}_t &= nL_t \end{aligned}$$

So far we assumed that the saving rate is constant. However, as we discussed in the context of the consumer decision problem, saving may not be proportional to current income, since consumers are also concerned about the future income. As Lucas (1976) criticize the problem of the reduced form estimation, constant saving rate may not be robust assumption. In that case, we can not trust any policy implication based on the constant saving rate. Moreover, we would like to develop a model which allows us to analyze the impact of policy change on real economy through a change in saving rate.

For this purposes we must endogenize the saving rate. For the sake of simple explanation, we would like to start with a discrete model. The discrete version of the above model is

$$\begin{aligned} K_{t+1} &= sF(K_t, T_t L_t) + (1-\delta) K_t \\ T_{t+1} &= (1+g) T_t \\ L_{t+1} &= (1+n) L_t \end{aligned}$$

We want to endogenize s. Since we know

$$\begin{aligned} C_t + S_t &= F(K_t, T_t L_t), \\ S_t &= sF(K_t, T_t L_t), \end{aligned}$$

$$sF(K_t, T_t L_t) = F(K_t, T_t L_t) - C_t.$$

That is,

$$
\begin{aligned}
K_{t+1} &= F(K_t, T_t L_t) + (1-\delta) K_t - C_t \qquad (1)\\
T_{t+1} &= (1+g) T_t \qquad (2)\\
L_{t+1} &= (1+n) L_t \qquad (3)
\end{aligned}
$$

In order to determine the saving rate, we must know how consumers decide their consumption. But we know that consumers are not only concerned about the current income, but also concerned about the future income. Hence, consumers' optimization problem must be dynamic. One of the extreme assumption is that a representative consumer maximize the following utility function given equations (1), (2) and (3).

$$\sum_{t=\tau}^{\infty} \beta^{(t-\tau)} U(C_t)$$

That is, this consumer behaves such as he never die and he maximizes the discounted utility from consumption.

The next chapter explain how to solve a dynamic optimization problem. Chapter 3 applies this method into the neoclassical growth model.

2 Mathematical Preparation

In this section, I explain the dynamic optimization. This method can be applied into the variety of economic problem. I, first, discuss a discrete model. Then I explain a continuos model. I explain it as simple as possible, and I do not discuss any technical problems. Hence, it can not be seen as proofs.

2.1 A discrete model (A Finite Horizon Problem)

Consider the following problem

$$
\begin{aligned}
& \max_{\{X_t\}} \left\{ \sum_{t=\tau}^{T-1} \beta^{(t-\tau)} r(X_t, S_t) + \beta^{(T-\tau)} V_T(S_T) \right\},\\
s.t.\ S_{t+1} &= G(X_t, S_t),\\
& S_\tau \text{ is given.}
\end{aligned}
$$

The variable, $\{X_t\}$, is called a control variable. It can be a vector. This is the variable which an agent tries to control in order to maximize his objective function. In the case of the neoclassical growth model, a representative agent chooses consumption, $\{C_t\}$. That is, the control variable in the neoclassical growth model is C_t. The variable, $\{S_t\}$, is called a state variable. It can be a vector. The state variable summarizes the current state of the economy. The vector $\{(K_t, T_t, L_t)\}$ is a state vector in the neoclassical growth model.

The function $r(X_t, S_{t+1})$ is called the one period return function. One period return function summarizes the reward from current state variable and the current control variable. In the case of the neoclassical growth model, $r(X_t, S_{t+1}) = U(C_t)$. The function $G(X_t, S_t)$ is called the transition function. It describes the dynamic behavior of the state variables. In the case of the neoclassical growth model

$$G(C_t, (K_t, T_t, L_t)) = \left\{ \begin{array}{c} F(K_t, T_t L_t) + (1-\delta) K_t - C_t \\ (1+g) T_t \\ (1+n) L_t \end{array} \right\}$$

The function, $V_T(S_T)$ is called the value function at the last period. Since the state variable summarizes the current state of economy. The value function indicates, how much value he expects to obtain in the future when the current state is S_T.

Given S_τ, when the agent chooses X_τ at date τ, $S_{\tau+1}$ is automatically determined. Given this S_{t+1}, the agent can choose $X_{\tau+1}$ at date $\tau+1$, and it determines $S_{\tau+2}$, and so on. In this way, the agent can recursively decide his decisions. Note that once the agent knows, S_t, he does not need to worry about other past variables $\{(X_s, S_s)\}_{s<t}$ since the state variable summarize every important information of the economy at that time. That is, the agent can ignore date $s < t$, when he makes his decision at date t. Hence, his maximization problem at date $T-1$ can be simplified as follows:

$$\begin{aligned} &\max_{X_{T-1}} \{r(X_{T-1}, S_{T-1}) + \beta V_T(G(X_{T-1}, S_{T-1}))\}, \\ &S_{T-1} \text{ is given.} \end{aligned}$$

For a simple explanation, I assume that r, V_T, and G is concave function and we can apply the first order approach to solve this problem. The moreover, the solution is interior. Then

$$r_1(X_{T-1}, S_{T-1}) + \beta V_T'(G(X_{T-1}, S_{T-1})) G_1(X_{T-1}, S_{T-1}) = 0.$$

Using an implicit function theorem, we can derive a policy function $x_{T-1}(.)$:

$$X_{T-1} = x_{T-1}(S_{T-1})$$

and this policy function must satisfy

$$r_1(x_{T-1}(S_{T-1}), S_{T-1}) + \beta V_T'(G(x_{T-1}(S_{T-1}), S_{T-1})) G_1(x_{T-1}(S_{T-1}), S_{T-1}) = 0.$$

That is, the optimal decision X_1 at date 1 is a function of S_1. Since the policy function is an optimal strategy, we can define the value function at date $T = 1$, $V_{T-1}(\cdot)$, as follows:

$$
\begin{aligned}
& \max_{X_{T-1}} \left\{ r\left(X_{T-1}, S_{T-1}\right) + \beta V_T\left(G\left(X_{T-1}, S_{T-1}\right)\right) \right\} \\
= \; & r\left(x_{T-1}\left(S_{T-1}\right), S_{T-1}\right) + \beta V_T\left(G\left(x_{T-1}\left(S_{T-1}\right), S_{T-1}\right)\right) \\
\equiv \; & V_{T-1}\left(S_{T-1}\right).
\end{aligned}
$$

That is, $V_{T-1}(S_{T-1})$ summarizes the expected present value of the discounted future reward stream at date $T-1$ and the current state is S_1. Note that envelope theorem implies that

$$
V'_{T-1}\left(S_{T-1}\right) = r_2\left(x_{T-1}\left(S_{T-1}\right), S_{T-1}\right) + \beta V'_T\left(G\left(X_{T-1}, S_{T-1}\right)\right) G_2\left(X_{T-1}, S_{T-1}\right). \quad (4)
$$

Using $V_{T-1}(S_{T-1})$ we can rewrite the agent's problem at date $T-2$ as follows:

$$
\begin{aligned}
& \max_{X_{T-2}} \left\{ r\left(X_{T-2}, S_{T-2}\right) + \beta V_{T-1}\left(G\left(X_{T-2}, S_{T-2}\right)\right) \right\}, \\
& S_{T-2} \textit{ is given.}
\end{aligned}
$$

This is the similar problem as before. Therefore, we can define $V_{T-2}(S_{T-2})$ and using $V_{T-2}(S_{T-2})$, we can describe the problem at date $T-3$ and so on. That is, in general, an original problem can be rewritten as the sequence of a simple static problem:

$$
V_t\left(S_t\right) \equiv \max_{X_t} \left\{ r\left(X_t, S_t\right) + \beta V_{t+1}\left(G\left(X_t, S_t\right)\right) \right\}, \quad (5)
$$

This equation is called the Bellman Equation. Hence, a finite horizon problem can be solved by a sequence of policy functions and value functions, $\{(x_t(\cdot), V_t(\cdot))\}_{t=0}^{T-1}$. For any initial state variable, S_0, the optimal policy function $x_0(S_0)$ determines X_0 and the transition function $G(X_0, S_0)$ determines S_1. Given this S_1 the optimal policy function and the transition function determines X_1 and S_2 and so on.

Let me interpret the first order conditions and the envelop conditions of equation (5). The first order conditions of equation (5) are

$$
0 = r_1\left(x_t\left(S_t\right), S_t\right) + \beta V'_{t+1}\left(G\left(x_t\left(S_t\right), S_t\right)\right) G_1\left(x_t\left(S_t\right), S_t\right), \quad (6)
$$

for any t. The first term of the right hand side of equation (6) is the marginal return from changing X_t. However, when the agent changes X_t, it affects not only the current return, but also the future returns by changing the future state variable. When the agent slightly changes X_t, it will change S_{t+1} by $G_1(X_t, S_t)$. If S_{t+1}slightly moves, it changes the present value of the future reward at date $t+1$ by $V'_{t+1}(S_{t+1})$. Since the agent discount the future by β, the future impact of changing X_t is $\beta V'_{t+1}(G(X_t, S_t)) G_1(X_t, S_t)$. If the agent optimally chooses variables, these two effects must be the same at any date t.

The envelope theorem implies

$$
V'_t\left(S_t\right) = r_2\left(x_t\left(S_t\right), S_t\right) + \beta V'_{t+1}\left(G\left(x_t\left(S_t\right), S_t\right)\right) G_2\left(x_t\left(S_t\right), S_t\right), \quad (7)
$$

for any t. It shows the component of the marginal benefit of changing the state variable, S_t. When the state variable changes a little bit, it changes the current return today by $r_2(x_t(S_t), S_t)$. But since changing the state variable at date t will change the future state variable S_{t+1} by $G_2(X_t, S_t)$, it also has a dynamic effect. Since changing S_{t+1} affects the present value of the future reword by $V'_{t+1}(S_{t+1})$ and the agent discounts the future by β, the total effect must be $\beta V'_{t+1}(G(X_t, S_t)) G_2(X_t, S_t)$.

Example: Consider the following the neoclassical growth model,

$$\max_{\{C_t\}} \sum_{t=\tau}^{T-1} \beta^{(t-\tau)} U(C_t) + \beta^{(T-\tau)} V_T(K_T, T_T, L_T),$$

$$\begin{aligned} s.t.\ K_{t+1} &= F(K_t, T_t L_t) + (1-\delta) K_t - C_t, \\ T_{t+1} &= (1+g) T_t, \\ L_{t+1} &= (1+n) L_t, \\ & \quad given\ \ (K_\tau, T_\tau, L_\tau) \end{aligned}$$

We can define the Bellman equation as follows:

$$V_t(K_t, T_t, L_t) = \max_{C_t} \{U(C_t) + \beta V_{t+1}(K_{t+1}, T_{t+1}, L_{t+1})\},$$

$$\begin{aligned} s.t.\ K_{t+1} &= F(K_t, T_t L_t) + (1-\delta) K_t - C_t, \\ T_{t+1} &= (1+g) T_t, \\ L_{t+1} &= (1+n) L_t, \end{aligned}$$

The first order condition is

$$U'(C_t) = \beta V_{t+1K}(K_{t+1}, T_{t+1}, L_{t+1})$$

and the envelope theorem implies

$$V_{tK}(K_t, T_t, L_t) = \beta V_{t+1K}(K_{t+1}, T_{t+1}, L_{t+1}) [F_K(K_t, T_t L_t) + (1-\delta)].$$

Combining the first order condition and the envelope theorem, we can derive Euler equation:

$$U'(C_t) = \beta U'(C_{t+1}) [F_K(K_{t+1}, T_{t+1} L_{t+1}) + (1-\delta)]$$

The right hand side is the benefit from current consumption. However an increase in current consumption lowers the amount of saving. Therefore, it lowers capital stock at date $t+1$. A reduction of capital stock lowers consumption at date $t+1$ by $F_K(K_{t+1}, T_{t+1} L_{t+1}) + (1-\delta)$. Since the agent discount the future by β, the marginal cost of increase in the current consumption at date t is $\beta U'(C_{t+1}) [F_K(K_{t+1}, T_{t+1} L_{t+1}) + (1-\delta)]$. The marginal benefit must be equal to the marginal cost. Another way to look at Euler equation is

$$\frac{\beta U'(C_{t+1})}{U'(C_t)} = \frac{1}{F_K(K_{t+1}, T_{t+1} L_{t+1}) + (1-\delta)}$$

The left hand side is the marginal rate of substitution between consumption at date t and $t+1$; the right hand side is the marginal rate of transformation from production function between consumption at date t and date $t+1$. These two values must be the same.

2.2 A Discrete Model (An Infinite Horizon Model)

Now let me consider the case, T goes infinite. That is, an original problem is

$$\begin{aligned} U(S_\tau, \tau) &= \max_{\{X_t\}} \left\{ \sum_{t=\tau}^{\infty} \beta^{(t-\tau)} r(X_t, S_t) \right\}, \\ s.t.\ S_{t+1} &= G(X_t, S_t), \\ & \quad S_\tau \text{ is given.} \end{aligned}$$

If T goes infinite, there is no last period and we can not use the previous method. However, we can explain it by analogy between this model and the previous one. Assume that a policy function of this original problem is $x_u(S_\tau, \tau)$.

Consider a following corresponding recursive problem:

$$\begin{aligned} V(S_t) &= \max_{X_t} \{ r(X_t, S_t) + \beta V(S_{t+1}) \}, \\ S_{t+1} &= G(X_t, S_t). \end{aligned} \tag{8}$$

With certain mild conditions (continuity and compactness), we can show that there exists a unique value function $V(\cdot)$ which satisfies equation (8). Moreover, it is shown that there exists a stable policy function, $x(S_t)$ and that with the further assumptions (strict concave), the policy function is unique. Finally, with further mild conditions, it is shown that $U(S_\tau, \tau) = V(S_\tau)$ and $x_u(S_\tau, \tau) = x(S_\tau)$. That is, it is shown that the Bellman equation (8) is equivalent to the original problem and the value function and policy function is time invariant.

An economic interpretation of the Bellman equation (8) is that the agent maximizes the sum of current return, $r(X_t, S_t)$, and the present value of the discounted future returns, $\beta V(S_{t+1})$. He is concerned about the trade off between the current benefits and the future benefits when he chooses X_t. Since he lives forever, it does not matter when he makes his decisions. Therefore, the value function and the policy function does not depend on time. His problem is stationary.

There are several methods to analyze the Bellman equation (8): the numerical method, the guess and verify method and Euler Equation. Let me explain these methods.

The Numerical method: Note that the original problem can be approximated by a finite horizon problem:

$$\begin{aligned} U(S_\tau,\tau) &= \max_{\{X_t\}}\left\{\lim_{T\to\infty}\left[\sum_{t=\tau}^{T-1}\beta^{(t-\tau)}r(X_t,S_t)+\beta^{(T-\tau)}V_T(S_T)\right]\right\}, \\ s.t.\ S_{t+1} &= G(X_t,S_t), \\ & S_\tau \ is\ given. \end{aligned}$$

Similarly, it is shown that we can approximate the Bellman equation (8) by the corresponding finite horizon problem.

$$\begin{aligned} V(S) &= \lim_{t\to-\infty} V_t(S). \\ V_t(S) &\equiv \max_X\{r(X,S)+\beta V_{t+1}(G(X,S))\} \end{aligned} \tag{9}$$

Since we know that the value function $V(\cdot)$ is unique, choose an arbitrary initial value function $V_T(\cdot)$. Then we can iterate equation (9) for any S by a computer. Since we know that there exists a value function $V(S)$, the iteration must converge.

Guess and Verify method: Another way to analyze the property of equation (8) is guess and very method. The first, guess what would be the property of $V(\cdot)$ and assume that the property is true. Then very that your guess satisfy equation (8). Since we know the value function is unique, if a property satisfies equation (8), the value function must have the property. For example, suppose that $r(X,S)$ and $G(X,S)$ are constant return to scale, then equation (8) can be rewritten as

$$\begin{aligned} V(S_t) &= \max_{X_t}\left\{r\left(\frac{X_t}{S_t},1\right)S_t+\beta V\left(G\left(\frac{X_t}{S_t},1\right)S_t\right)\right\}, \\ &= \max_{g_t}\{r(g_t,1)S_t+\beta V(G(g_t,1)S_t)\}. \end{aligned}$$

Now, let me guess that the value function is linear in S_t: that is, there exists q such that $V(S_t)=qS_t$. Then

$$\begin{aligned} & \max_{g_t}\{r(g_t,1)S_t+\beta V(G(g_t,1)S_t)\} \\ =& \max_{g_t}\{r(g_t,1)S_t+\beta qG(g_t,1)S_t\} \\ =& \max_{g_t}\{r(g_t,1)+\beta qG(g_t,1)\}S_t, \end{aligned}$$

Hence, if

$$q=\max_{g_t}\{r(g_t,1)+\beta qG(g_t,1)\},$$

then my guess is correct. Suppose that I find q^* which satisfies the above equation. Then I can verify my guess. Since the value function is unique, this must be the property of this value function.

Consider another example. Since many applied economists restrict their attention to the case to which we can apply the first order condition, I explain this method by using the first order condition. The policy function $x(\cdot)$ must satisfy the first order condition and the bellman equation for any S as follows:

$$\begin{aligned} 0 &= r_1(x(S_t), S_t) + \beta V'(G(x(S_t), S_t))\, G_1(x(S_t), S_t), && (10) \\ V(S_t) &= r(x(S_t), S_t) + \beta V(G(x(S_t), S_t)). && (11) \end{aligned}$$

Given the value function $V(\cdot)$, equation (10) determines $x(\cdot)$; given $x(\cdot)$, equation (11) determines $V(\cdot)$. That is, these are simultaneous equations. We can analyze our model by examining these two equations.

Example: In general we can not solve a closed form solution. However, there is a special case in which we can find a closed form solution. A special case is $r(X, S)$ is quadratic and $G(X, S)$ is linear. Then it is well known that we can solve a closed form solution. Consider the following investment problem. Suppose that $r(I, K) = zK - pI - \frac{A}{2}I^2$ and $G(I, S) = I + (1-\delta)K$, where K is capital stock, I is investment, p is the investment price, z is the instantaneous return on capital and A is the parameter of adjustment cost. Then the Bellman equation is

$$\begin{aligned} V(K_t) &= \max_{I_t} \left\{ zK_t - pI_t - \frac{A}{2}I_t^2 + \beta V(K_{t+1}) \right\}, && (12) \\ K_{t+1} &= I_t + (1-\delta)K_t. \end{aligned}$$

Using the first order conditions

$$p + AI_t = \beta V'(K_{t+1}).$$

Guess that $V(K) = a + bK_t + \frac{c}{2}K_t^2$. Then $V'(K) = b + cK$. Then

$$\begin{aligned} p + AI_t &= \beta \left[b + c\left(I_t + (1-\delta)K_t\right) \right], \\ I_t &= D + FK_t, \\ \textit{where } D &= \frac{-p + \beta b}{A - \beta c}, \\ F &= \frac{\beta c(1-\delta)}{A - \beta c}. \end{aligned}$$

Then

$$
\begin{aligned}
& \max_{I_t}\left\{zK_t - pI_t - \frac{A}{2}I_t^2 + \beta V\left(K_{t+1}\right)\right\} \\
= & \ zK_t - p\left(D+FK_t\right) - \frac{A}{2}\left(D+FK_t\right)^2 + \beta V\left(D+FK_t+(1-\delta)K_t\right) \\
= & \ zK_t - p\left(D+FK_t\right) - \frac{A}{2}\left(D+FK_t\right)^2 + \\
& \beta a + \beta b\left[D+\left(F+(1-\delta)\right)K_t\right] + \beta\frac{c}{2}\left[D+\left(F+(1-\delta)\right)K_t\right]^2 \\
= & \ -pD - \frac{A}{2}D^2 + \beta a + \beta bD + \beta\frac{c}{2}D^2 \\
& + \left[z - pF - ADF + \beta b\left(F+(1-\delta)\right) + \beta cD\left(F+(1-\delta)\right)\right]K_t \\
& + \left[-\frac{A}{2}F^2 + \beta\frac{c}{2}\left(F+(1-\delta)\right)^2\right]K_t^2.
\end{aligned}
$$

If my guess is correct for any K_t, then

$$
\begin{aligned}
a &= -pD - \frac{A}{2}D^2 + \beta a + \beta bD + \beta\frac{c}{2}D^2 \\
b &= z - pF - ADF + \beta b\left(F+(1-\delta)\right) + \beta cD\left(F+(1-\delta)\right) \\
c &= -\frac{A}{2}F^2 + \beta\frac{c}{2}\left(F+(1-\delta)\right)^2 \\
\text{where } D &= \frac{-p+\beta b}{A-\beta c} \\
F &= \frac{\beta c(1-\delta)}{A-\beta c}
\end{aligned}
$$

We have three unknowns, a, b and c, and three equations. Hence, we can solve these equations. Once we find a, b and c, we can derive the value function and the policy function.
Homework: Find a, b and c, and derive the value function and the policy function.
Homework: Consider the following growth model:

$$
\begin{aligned}
v\left(k_0\right) &= \max_{\{c_t\}}\sum_{t=0}^{\infty}\beta^t\log c_t,\ 0<\beta<1 \\
s.t.\ k_{t+1} &= Ak_t^{\alpha} - c_t,\ 0<\alpha<1, \\
& \quad k_0 \text{ is given.}
\end{aligned}
$$

Show that

$$
v(k) = (1-\beta)^{-1}\left[\log A(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\log A\beta\alpha\right] + \frac{\alpha}{1-\alpha\beta}\log k.
$$

As you can see that the guess and verify method demands many calculations. If we are only interested in a policy function, but not a value function, there is one way to reduce calculation: the use of envelop theorem. The first order condition and the envelop theorem of the Bellman equation (8) imply that

$$\begin{aligned} 0 &= r_1 (x(S_t), S_t) + \beta V' (G(x(S_t), S_t)) G_1 (x(S_t), S_t), \quad (13) \\ V'(S_t) &= r_2 (x(S_t), S_t) + \beta V' (G(x(S_t), S_t)) G_2 (x(S_t), S_t), \quad (14) \end{aligned}$$

for any S_t. Again two unknown functions $x(\cdot)$ and $V'(\cdot)$ can be solved by two equations. Note that the first order condition (13) is the same as the first order condition (10); the envelop condition (14) is not the same as the Bellman equation (11). Also note that the previous two equations determine $x(\cdot)$ and $V(\cdot)$; the current two conditions determine $x(\cdot)$ and $V'(\cdot)$. In general, there exists C such that

$$V(S) = \int V'(S)\, dS + C.$$

Since equation (13) and (14) can not determine C, without a boundary condition, equation (13) and (14) can not solve the value function. However, if we are only interested in a policy function, equations (13) and (14) can solve it.

Example: Let me apply the envelop theorem to the previous investment model. The envelop theorem for equation (12) implies that

$$V'(K_t) = z + \beta (1-\delta) V'(K_{t+1}).$$

My guess was $V'(K) = b + cK$ and this guess implied that the policy function is affine in K_t, $I_t = D + FK_t$. Therefore,

$$\begin{aligned} & z + \beta (1-\delta) V'(K_{t+1}) \\ = & z + \beta (1-\delta) \{b + c[I_t + (1-\delta) K_t]\} \\ = & z + \beta (1-\delta) \{b + c[D + FK_t + (1-\delta) K_t]\} \\ = & z + \beta (1-\delta)(b + cD) + \beta (1-\delta) c [F + (1-\delta)] K_t \end{aligned}$$

therefore, if my guess is correct, the following two equations must be satisfied.

$$\begin{aligned} b &= z + \beta (1-\delta)(b + cD) \\ c &= \beta (1-\delta) c [F + (1-\delta)] \end{aligned}$$

Unknown variables are two, b and c, hence we can solve equations and derive a policy function.

Homework: Solve b and c. Derive a policy function. Check that your answer is the same as the previous your answer.

Euler Equation: In general, it is difficult to solve a closed form solution. However, it is possible to analyze the property of solutions for more general class of functions. The derivation of the Euler equation is the most famous one. In general, if you combine the first order condition (13) and the envelop condition (14), we can eliminate $V'(S)$ and derive the first order nonlinear difference equation:

$$\frac{r_1(X_t, S_t)}{\beta G_1(X_t, S_t)} = \frac{r_1(X_{t+1}, S_{t+1})}{G_1(X_{t+1}, S_{t+1})} G_2(X_{t+1}, S_{t+1}) - r_2(X_{t+1}, S_{t+1}). \tag{15}$$

Together with the transition equation,

$$S_{t+1} = G(X_t, S_t),$$

we can solve the sequence of X_t and S_t. However, there is a difficulty. Although we have one initial condition, S_τ, we need one more boundary condition to solve two difference equations. In a finite horizon case, since the value function at the end period is given, the first order condition,

$$r_1(X_{T-1}, S_{T-1}) + \beta V_T'(G(X_{T-1}, S_{T-1})) G_1(X_{T-1}, S_{T-1}) = 0,$$

serves as the boundary condition. The difficult question is what would be the appropriate boundary condition in an infinite horizon problem. It is shown that the following transversality condition is sufficient for an optimal solution:

$$\lim_{t\to\infty} \beta^t V'(S_t) S_t = -\lim_{t\to\infty} \beta^{t-1} \frac{r_1(X_{t-1}, S_{t-1})}{G_1(X_{t-1}, S_{t-1})} S_t = 0. \tag{16}$$

Since $V'(S_t)$ is the marginal value of S_t, $\beta^t V'(S_t) S_t$ is the present value of stock at t. The transversality condition implies that when t goes infinite, the present value of stock must be negligible. That is, we should not be concerned about an infinite period later.

Note that although the first order condition (13) and the envelop condition (14) can derive the Euler equation (15), the Euler equation (15) can not derive the first order condition (13) and the envelop condition (14). That is, the elimination of $V'(S_t)$ discards useful information. Therefore, the Euler equation is a necessary condition for the original problem, but not sufficient. However, given technical usual conditions, concavity etc, it is known that the Euler equation and the transversality condition are sufficient.

Example: Let me derive the Euler equation of the previous investment problem. From the first order condition and the envelop condition of the previous investment model, the Euler equation is

$$p + AI_t = z + \beta(1-\delta)(p + AI_{t+1}),$$

and the transition equation is

$$K_{t+1} = I_t + (1-\delta) K_t.$$

The initial condition K_0 and the transversality condition is

$$\lim_{t\to\infty} \beta^{t-1} (p + AI_{t-1}) K_t = A \lim_{t\to\infty} \beta^{t-1} I_{t-1} K_t = 0.$$

Example: Consider the previous neoclassical growth model:

$$\begin{aligned} & \max_{\{C_t\}} \sum_{t=\tau}^{\infty} \beta^{(t-\tau)} U(C_t), \\ s.t.\ K_{t+1} &= F(K_t, T_t L_t) + (1-\delta) K_t - C_t, \\ T_{t+1} &= (1+g) T_t, \\ L_{t+1} &= (1+n) L_t, \\ & \textit{given } (K_\tau, T_\tau, L_\tau) \end{aligned}$$

We can define the Bellman equation as follows:

$$\begin{aligned} V(K_t, T_t, L_t) &= \max_{C_t} \{U(C_t) + \beta V(K_{t+1}, T_{t+1}, L_{t+1})\}, \\ s.t.\ K_{t+1} &= F(K_t, T_t L_t) + (1-\delta) K_t - C_t, \\ T_{t+1} &= (1+g) T_t, \\ L_{t+1} &= (1+n) L_t, \end{aligned}$$

The first order condition is

$$U'(C_t) = \beta V_K (K_{t+1}, T_{t+1}, L_{t+1})$$

and the envelope theorem implies

$$V_K (K_t, T_t, L_t) = \beta V_K (K_{t+1}, T_{t+1}, L_{t+1}) [F_K (K_t, T_t L_t) + (1-\delta)].$$

Combining the first order condition and the envelope theorem, we can derive the same Euler equation as before:

$$U'(C_t) = \beta U'(C_{t+1}) [F_K (K_{t+1}, T_{t+1} L_{t+1}) + (1-\delta)].$$

One additional condition is the transversality condition:

$$\lim_{t\to\infty} \beta^{t-1} U'(C_{t-1}) K_t = 0.$$

An economic interpretation of this transversality condition is that it is not optimal to keep capital stock at the final date when the marginal value of consumption is positive. If so, the agent can always lowers capital stock and consume more.

2.3 A continuous infinite horizon model

Now I want to analyze a continuous model. Using the previous discrete model, I would like to derive the corresponding continuous model as the limit of the previous discrete model. In the previous model, time goes like t, $t+1$, $t+2$. In this section, I assume that time goes like t, $t+\Delta$, $t+2\Delta$....Then I take the limit of Δ. I assume that one period return between t and $t+\Delta$ is constant, $\Delta r\left(X_t, S_t\right)$. Therefore, the sum of the returns is

$$\Delta r\left(X_\tau, S_\tau\right)+\beta\Delta r\left(X_{\tau+\Delta}, S_{\tau+\Delta}\right)+\beta^2\Delta r\left(X_{\tau+2\Delta}, S_{\tau+3\Delta}\right)+...,$$

where τ is an initial period. We can define at time t by $t_j=\tau+j\Delta$. Hence, $j=\frac{t_j-\tau}{\Delta}$. Therefore, the above sum can be rewritten as

$$\Delta r\left(X_\tau, S_\tau\right)+\beta^{\frac{t_1-\tau}{\Delta}}\Delta r\left(X_{t_1}, S_{t_1}\right)+\beta^{\frac{t_2-\tau}{\Delta}}\Delta r\left(X_{t_2}, S_{t_2}\right)...,$$

Assume that

$$\beta=\frac{1}{1+\Delta\rho},$$

where ρ is the discount rate. Then the continuous version of the original model can be expressed as

$$\begin{aligned} U\left(S_\tau,\tau\right) &= \max_{\{X_t\}}\left\{\lim_{\Delta\to 0}\sum_{j=0}^{\infty}\left(\frac{1}{1+\Delta\rho}\right)^{\frac{\left(t_j-\tau\right)}{\Delta}}\Delta r\left(X_{t_j}, S_{t_j}\right)\right\}, \\ s.t.\ S_{t+\Delta} &= G\left(X_t, S_t\right)=\Delta G^c\left(X_t, S_t\right)+S_t, \\ & \quad S_\tau \textit{ is given.} \end{aligned}$$

Since

$$e^{-\rho t}=\lim_{\Delta\to 0}\left[\frac{1}{1+\Delta\rho}\right]^{\frac{t}{\Delta}},$$

the continuous version of the previous model is written as

$$\begin{aligned} U\left(S_\tau,\tau\right) &= \int_\tau^\infty e^{-\rho(t-\tau)}r\left(X_t, S_t\right)dt \\ s.t.\ \dot{S}_t &= G^c\left(X_t, S_t\right) \\ & \quad S_\tau \textit{ is given.} \end{aligned}$$

I would like to derive the continuous version of the Bellman equation as the limit of the previous discrete model. Since $\beta=\frac{1}{1+\Delta\rho}$

$$\begin{aligned} V\left(S_t\right) &= \max_{X_t}\left\{\Delta r\left(X_t, S_t\right)+\frac{1}{1+\Delta\rho}V\left(S_{t+\Delta}\right)\right\}, \\ S_{t+\Delta}-S_t &= \Delta G^c\left(X_t, S_t\right). \end{aligned}$$

We can modify the value function as follows.

$$
\begin{aligned}
(1+\Delta\rho)V(S_t) &= \max_{X_t}\{(1+\Delta\rho)\Delta r(X_t,S_t)+V(S_{t+\Delta})\} \\
\Delta\rho V(S_t) &= \max_{X_t}\{\Delta r(X_t,S_t)+\Delta^2\rho r(X_t,S_t)+[V(S_{t+\Delta})-V(S_t)]\} \\
\rho V(S_t) &= \max_{X_t}\left\{r(X_t,S_t)+\Delta\rho r(X_t,S_t)+\frac{V(S_{t+\Delta})-V(S_t)}{\Delta}\right\}
\end{aligned}
$$

When Δ goes 0, the continuous version of the Bellman equation is derived as follows.

$$
\begin{aligned}
\rho V(S_t) &= \max_{X_t}\left\{r(X_t,S_t)+V'(S_t)\dot{S}_t\right\} \\
&= \max_{X_t}\{r(X_t,S_t)+V'(S_t)G^c(X_t,S_t)\}
\end{aligned}
$$

Let me analyze the continuous version of the Bellman equation. The first order condition of this Bellman equation is

$$0=r_1(x(S_t),S_t)+V'(S_t)G_1^c(x(S_t),S_t). \tag{17}$$

Substitute the policy function into the Bellman equation,

$$\rho V(S_t)=r(x(S_t),S_t)+V'(S_t)G^c(x(S_t),S_t). \tag{18}$$

Again, the first order condition (17) and the Bellman equation (18) solves the value function $V(\cdot)$ and $x(\cdot)$. We can apply the Guess and verified method to solve two equations.

Optimal Control: The more applicable and popular method to solve the above equations are the use of the Necessary conditions. The envelop theorem implies

$$\rho V'(S_t)=r_2(X_t,S_t)+V''(S_t)G^c(X_t,S_t)+V'(S_t)G_2^c(X_t,S_t).$$

Note that $V''(S_t)G^c(X_t,S_t)=V''(S_t)\dot{S}_t=\frac{dV'(S_t)}{dt}$. Hence the above equation implies

$$\frac{dV'(S_t)}{dt}=\rho V'(S_t)-[r_2(X_t,S_t)+V'(S_t)G_2^c(X_t,S_t)]. \tag{19}$$

Define the costate variable, λ_t,

$$\lambda_t=V'(S_t).$$

Then the first order condition (17) and the envelop theorem (19) and the transition equation can be rewritten as

$$
\begin{aligned}
0 &= r_1(x(S_t),S_t)+\lambda_t G_1^c(x(S_t),S_t), && (20)\\
\dot{\lambda}_t &= \rho\lambda_t-[r_2(x(S_t),S_t)+\lambda_t G_2^c(x(S_t),S_t)], && (21)\\
\dot{S}_t &= G^c(x(S_t),S_t),\ S_\tau \textit{ is given.} && (22)
\end{aligned}
$$

The first order condition (20) determines a policy function $x(S_t)$ given λ_t. Given the policy function $x(S_t)$, equations (21) and (22) determine the path of the costate variable, λ_t and the state variable, S_t. Note that since we have one boundary condition, S_τ, we need an additional boundary condition to solve the equations (21) and (22). For this purposes, we need the following transversality condition:

$$\lim_{t\to\infty} e^{-\rho t} V'(S_t) S_t = \lim_{t\to\infty} e^{-\rho t} \lambda_t S_t = 0. \tag{23}$$

When we define the costate variable, $\lambda_t = V'(S_t)$, we lose important information. Therefore, although we can derive equation (21) from equation (19), the opposite direction is not true. Therefore, equations (20), (21) and (19) are the necessary conditions for the original problem, but not sufficient. However, given usual technical assumptions, concavity etc, it is shown that equations (20), (21),e (19) and (23) are sufficient conditions for the original problem.

The above conditions are nicely summarized by defining Hamiltonian $H(X_t, S_t, \lambda_t)$:

$$H(X_t, S_t, \lambda_t) = r(X_t, S_t) + \lambda_t G^c(X_t, S_t).$$

Then above conditions are expressed as

$$\begin{aligned} 0 &= H_X(X_t, S_t, \lambda_t) \\ \frac{d\left[\lambda_t e^{-\rho t}\right]}{dt} &= -e^{-\rho t} H_S(X_t, S_t, \lambda_t) \\ \dot{S}_t &= H_\lambda(X_t, S_t, \lambda_t),\ S_\tau \textit{ is given.} \\ 0 &= \lim_{t\to\infty} e^{-\rho t} \lambda_t S_t \end{aligned}$$

These are the useful conditions for analyzing the continuous version of the dynamic optimization problem.

Hamiltonian can be nicely interpreted. When we choose a dynamic optimization problem, we know that current decisions affects not only the current return, but also the future returns. The second term of Hamiltonian summarizes the impact on the future returns. Remember, $\lambda_t = V'(S_t)$. That is, the costate variable can be interpreted as the marginal impact of the state variable on the present value of the discounted future returns. Since $\dot{S}_t = G^c(X_t, S_t)$, the second term is interpreted as the impact of a change in S_t on the future returns. Therefore, Hamiltonian summarizes the important trade off of the dynamic optimization problem: the impact on the current return and the future returns. The first order condition, $0 = H_X$, implies that the agent chooses the control variable X_t such as he maximizes Hamiltonian.

3 Representative Agent Model

In this section I apply optimal control to the neoclassical growth model and analyze the neoclassical growth model. First, let me formulate the continuous version of the neoclassical growth model.

$$
\begin{aligned}
& \max_{C_t} \int_0^\infty e^{-\rho t} U\left(\frac{C_t}{L_t}\right) dt \\
\dot{K}_t &= F\left(K_t, T_t L_t\right) - \delta K_t - C_t, \\
\dot{T}_t &= g T_t, \\
\dot{L}_t &= n L_t,
\end{aligned}
$$

where K_0, T_0 and L_0 are given. In order to simplify the problem, let me define capital stock per efficiency unit, k_t, and consumption per efficiency unit, c_t;

$$
\begin{aligned}
k_t &= \frac{K_t}{T_t L_t}, \\
c_t &= \frac{C_t}{T_t L_t}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\frac{\dot{k}_t}{k_t} &= \frac{\dot{K}_t}{K_t} - \left(\frac{\dot{T}_t}{T_t} + \frac{\dot{L}_t}{L_t}\right), \\
&= \frac{F\left(K_t, H_t\right) - \delta K_t - C_t}{K_t} - \left(\frac{\dot{T}_t}{T_t} + \frac{\dot{L}_t}{L_t}\right), \\
&= \frac{\left[F\left(k_t, 1\right) - \delta k_t - c_t\right] T_t L_t}{K_t} - (g+n), \\
&= \frac{\left[f\left(k_t\right) - \delta k_t - c_t\right]}{k_t} - (g+n), \\
\dot{k}_t &= f\left(k_t\right) - c_t - (g+n+\delta) k_t,
\end{aligned}
$$

where $f\left(k_t\right) = F\left(k_t, 1\right)$. From the second equation to the third, I use the assumption on the production function, the constant return to scale. Using T_t, k_t and c_t, I can rewrite the original problem as

$$
\begin{aligned}
& \int_0^\infty e^{-\rho t} U\left(c_t T_t\right) dt, \\
\dot{k}_t &= f\left(k_t\right) - c_t - (g+n+\delta) k_t, \\
\dot{T}_t &= g T_t,
\end{aligned}
$$

where k_0 and T_0 are given. Define Hamiltonian of this problem:

$$
H\left(c_t, k_t, H_t, \lambda_t, \mu_t\right) = U\left(c_t T_t\right) + \lambda_t \left[f\left(k_t\right) - c_t - (g+n+\delta) k_t\right] + \mu_t g T_t
$$

The first order conditions are

$$\begin{aligned}
\lambda_t &= U'(c_tT_t)\,T_t \\
\dot{\lambda}_t &= \rho\lambda_t - \lambda_t\left[f'(k_t) - (g+n+\delta)\right],\ 0 = \lim_{t\to\infty}\lambda_t k_t e^{-\rho t} \\
\dot{\mu}_t &= \rho\mu_t - U'(c_tT_t)\,c_t - \mu_t g,\ 0 = \lim_{t\to\infty}\mu_t T_t e^{-\rho t} \\
\dot{k}_t &= f(k_t) - c_t - (g+n+\delta)\,k_t,\ k_0 \ is\ given \\
\dot{T}_t &= gT_t,\ T_0\ is\ given
\end{aligned}$$

$$\begin{aligned}
\dot{\lambda}_t &= U'(c_tT_t)\,\dot{T}_t + U''(c_tT_t)\,T_t\left[c_t\dot{T}_t + \dot{c}_tT_t\right] \\
\frac{\dot{\lambda}_t}{\lambda_t} &= \frac{\dot{T}_t}{T_t} + \frac{U''(c_tT_t)}{U'(c_tT_t)}\left[c_t\dot{T}_t + \dot{c}_tT_t\right] \\
&= g + \frac{U''(c_tT_t)\,c_tT_t}{U'(c_tT_t)}\left[g + \frac{\dot{c}_t}{c_t}\right]
\end{aligned}$$

$$\begin{aligned}
g + \frac{U''(c_tT_t)\,c_tT_t}{U'(c_tT_t)}\left[g + \frac{\dot{c}_t}{c_t}\right] &= \rho - f'(k_t) + (g+n+\delta) \\
\frac{U''(c_tT_t)\,c_tT_t}{U'(c_tT_t)}\left[g + \frac{\dot{c}_t}{c_t}\right] &= \rho + \delta + n - f'(k_t) \\
\frac{\dot{c}_t}{c_t} &= \frac{1}{\theta(c_tT_t)}\left[f'(k_t) - (\rho+\delta+n)\right] - g \qquad (24) \\
wehre\ \theta(c_tT_t) &= -\frac{U''(c_tT_t)\,c_tT_t}{U'(c_tT_t)}
\end{aligned}$$

Equation (24) is the continuous version of Euler equation. The value, $\theta(c_tT_t)$, is called the coefficient of the relative risk aversion at c_tT_t. Roughly speaking, this coefficient measures the curvature of the agent's utility function and it capture the agent's risk attitude. In particular, when the coefficient decreases as c_tT_t increases, I say that the agent becomes less risk averse with regard to gambles that are proportional to c_tT_t as c_tT_t increases.

Homework: Read any microeconomics textbook and check the property of the coefficient of the relative risk aversion.

It is also known that $\frac{1}{\theta(c_tT_t)}$ measures the elasticity of substitution between consumption at any two points in time. To see this, consider the following two period problem. Suppose that $U(C_t,\ C_{t+1}) = u(C_t) + \beta u(C_{t+1})$ and that the budget constraint is $p_t C_t + p_{t+1} C_{t+1} = W$, where p_t and p_{t+1} are prices at t and $t+1$, and W is the wealth. Since the first corder condition is

$$\begin{aligned}\frac{p_{t+1}}{p_t} &= \frac{U_2(C_t, C_{t+1})}{U_1(C_t, C_{t+1})} \\ &= \beta\frac{u'(C_{t+1})}{u'(C_t)},\end{aligned}$$

It means that the elasticity of substitution, $-\frac{d\log\left(\frac{C_{t+1}}{C_t}\right)}{d\log\left(\frac{p_{t+1}}{p_t}\right)}$, must satisfy the following condition.

$$-\frac{d\log\left(\frac{C_{t+1}}{C_t}\right)}{d\log\left(\frac{p_{t+1}}{p_t}\right)} = -\frac{d\log\left(\frac{C_{t+1}}{C_t}\right)}{d\log\left(\delta\frac{u'(C_{t+1})}{u'(C_t)}\right)}$$

It is shown that

$$\begin{aligned}-\frac{d\log\left(\frac{C_{t+1}}{C_t}\right)}{d\log\left(\delta\frac{u'(C_{t+1})}{u'(C_t)}\right)} &= -\frac{d\left[\log(C_{t+1}) - \log(C_t)\right]}{d\left[\log(u'(C_{t+1})) - \log u'(C_t)\right]} \\ &\cong -\frac{d\left[\log(C_{t+1}) - \log(C_t)\right]}{d\left\{\frac{u''(C_t)C_t}{u'(C_t)}\left[\log(C_{t+1}) - \log(C_t)\right]\right\}} \\ &\cong \frac{1}{\theta(c_t T_t)}\ if\ C_{t+1} \cong C_t\end{aligned}$$

In order to make my analysis simpler, I assume that the coefficient of the relative risk aversion is constant.

$$-\frac{U''(c_t T_t)\, c_t T_t}{U'(c_t T_t)} = \theta.$$

It is well-known that the following function satisfy the above condition:

$$\begin{aligned}U(c_t T_t) &= \frac{(c_t T_t)^{(1-\theta)} - 1}{1-\theta},\ if\ \theta \neq 1, \\ &= \log(c_t T_t),\ if\ \theta = 1.\end{aligned}$$

This utility function is called the constant relative risk aversion utility function.

Homework: Check that the constant risk aversion utility function has the constant coefficient of the relative risk aversion.

Given the constant relative risk aversion utility function, Euler equation (24) becomes

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}\left[f'(k_t) - (\rho + \delta + n + \theta g)\right].$$

The corresponding transversality condition is

$$\begin{aligned} 0 &= \lim_{t\to\infty} \lambda_t k_t e^{-\rho t} \\ &= \lim_{t\to\infty} (c_t T_t)^{-\theta} T_t k_t e^{-\rho t} \\ &= \lim_{t\to\infty} (c_t)^{-\theta} e^{(1-\theta)gt} k_t e^{-\rho t} \\ &= \lim_{t\to\infty} (c_t)^{-\theta} k_t e^{-[\rho-(1-\theta)g]t}. \end{aligned}$$

Together with the transition equation of k_t, the following two differential equations and two boundary conditions solve the optimal growth path:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}\left[f'(k_t) - (\rho + \delta + n + \theta g)\right], \tag{25}$$

$$\dot{k}_t = f(k_t) - c_t - (g + n + \delta) k_t, \tag{26}$$

$$0 = \lim_{t\to\infty} (c_t)^{-\theta} k_t e^{-[\rho-(1-\theta)g]t},\ k_0 \textit{ is given.} \tag{27}$$

The Phase Diagram: One of the merits of working with a continuous model is that we can use the phase diagram. Let me describe what the phase diagram is. The first, let me define the steady state.

Definition 1 *On the steady state the path of* (c_t^*, k_t^*) *satisfies*

$$\frac{\dot{c}_t^*}{c_t^*} = \frac{\dot{k}_t^*}{k_t^*} = 0.$$

Therefore, the following equation must be satisfied on the steady sate:

$$\frac{\dot{c}_t^*}{c_t^*} = 0 : f'(k_t^*) = \rho + \delta + n + \theta g \tag{28}$$

$$\frac{\dot{k}_t^*}{k_t^*} = 0 : c_t^* = f(k_t^*) - (g + n + \delta) k_t^* \tag{29}$$

Consider the $c - k$ plain. The first, let me examine equation (28). Note that k_t^* is uniquely determined by equation (28). This is the vertical line on the $c - k$ plain. When $k_t = k_t^*$, $\frac{\dot{c}_t^*}{c_t^*} = 0$. When $k < k_t^*$, $f'(k_t) > \rho + \delta + n + \theta g$. Therefore, $\frac{\dot{c}_t}{c_t} > 0$. On the other hand, when $k > k_t^*$, $f'(k_t) < \rho + \delta + n + \theta g$. Therefore, $\frac{\dot{c}_t}{c_t} < 0$.

The next, look at equation (29). Note that on the steady state

$$\frac{dc_t^*}{dk_t^*} = f'(k_t^*) - (g+n+\delta)$$
$$\frac{d^2c_t^*}{d(k_t^*)^2} = f''(k_t^*) < 0$$

Since $\frac{d^2c_t^*}{d(k_t^*)^2} < 0$, c_t^* has the maximum value at

$$f'\left(k^{GR}\right) = g+n+\delta. \tag{30}$$

Capital stock, k^{GR} is called the golden rule level of the capital stock. If we want to maximize the steady state consumption per capita, equation (30) must be satisfied. When $k_t^* < k^{GR}$, $\frac{dc_t^*}{dk_t^*} > 0$ and $k_t^* > k^{GR}$, $\frac{dc_t^*}{dk_t^*} < 0$.

Compare equation (28) and equation (30), and note that

$$g+n+\delta < \rho+\delta+n+\theta g \; iff \; \rho > (1-\theta)g.$$

If the steady state satisfies the transversality condition (27), $\rho > (1-\theta)g$ must be true, which I assume now. It implies $f'(k_t^*) > f'\left(k_t^{GR}\right)$. Therefore, $k_t^* < k_t^{GR}$. Since the agent discount future, it is not optimal to reduce current consumption to reach the golden rule level of the capital stock. The steady state value of capital stock, k_t^*, is called the modified golden rule level of capital stock.

When $c_t > f(k_t) - (g+n+\delta)k_t$, $\dot{k}_t < 0$. When $c_t < f(k_t) - (g+n+\delta)k_t$, $\dot{k}_t > 0$. Therefore, putting two equations (28) and (29) on the $c-k$ plain, we can describe the following the Phase Diagram:

THE FIGURE 1 MUST BE HERE.

Look at the Phase Diagram, it describes the potential path of two differential equations. Let me consider three potential paths. Note that the phase diagram says that given k_0, there exists $c(k_0)$ which converges to the steady state. If $c_0 = c(k_0)$, economy will eventually reach the steady state. Since I assume $\rho > (1-\theta)g$, the steady state satisfies the transversality condition (27). Therefore, this path is optimal.

What if $c_0 > c(k_0)$, it is known that it hits $k_t = 0$ line in finite time. But when $k_t = 0$, c_t must jump to 0 from equation (26). Otherwise, k_t goes negative. However, the Euler equation (25) does not allow the jump. Hence this path violates the Euler equation (25).

Finally, what if $c_0 < c(k_0)$, the phase diagram says that this path eventually converges to the point $c_t = 0$ and $k_t < \infty$. It is known that this violate the transversality condition (27).

To see this, because k eventurally becomes larger than k^{GR}, the growth rate of consumption becomes

$$\begin{aligned}
\frac{\dot{c}_t}{c_t} &= \frac{1}{\theta}\left[f'(k_t) - (\rho + \delta + n + \theta g)\right] \\
&< \frac{1}{\theta}\left[g + n + \delta - (\rho + \delta + n + \theta g)\right] \\
&= \frac{1}{\theta}\left[(1-\theta)\, g - \rho\right]
\end{aligned}$$

Hence, for large t

$$c_t < c_0 e^{\frac{1}{\theta}[(1-\theta)g-\rho]t}$$

$$\begin{aligned}
& (c_t)^{-\theta}\, k_t e^{-[\rho-(1-\theta)g]t} \\
> & \left(c_0 e^{\frac{1}{\theta}[(1-\theta)g-\rho]t}\right)^{-\theta} k_t e^{-[\rho-(1-\theta)g]t} \\
= & c_0 e^{[\rho-(1-\theta)g]t} k_t e^{-[\rho-(1-\theta)g]t} \\
= & c_0 k_t > 0
\end{aligned}$$

Hence

$$0 < \lim_{t\to\infty} (c_t)^{-\theta}\, k_t e^{-[\rho-(1-\theta)g]t}.$$

This violates the transversality condition.

In sum, the unique optimal path is characterized by the path which converges to the steady state. For any given k_0, the agent chooses $c(k_0)$ expecting that it converges to the steady state. This path is called the saddle path.

Steady State Analysis: The above phase diagram suggests that economy will eventually converge to the steady state. Hence, it is reasonable to assume that the real economy locates near the steady state, and analyze what is the property of the steady state. On the steady state, c_t^* and k_t^* are constant. Since $y_t^* = f(k_t^*)$, y_t^* is also constant. Define the gross saving rate, s_t:

$$\begin{aligned}
s_t &= \frac{Y_t - C_t}{Y_t}, \\
&= \frac{y_t^* - c_t^*}{y_t^*} = s.
\end{aligned}$$

That is, on the steady state, the saving rate is constant. This is the assumption that the Solow model assumes. Therefore, it is derived from equation (29) that

$$\begin{aligned}
(g + n + \delta)\, k_t^* &= f(k_t^*) - c_t^*, \\
&= s f(k_t^*).
\end{aligned}$$

This is the steady state condition of the Solow model. Hence, the properties of the Solow model are maintained on the steady state. In fact, since c_t^*, y_t^* and k_t^* are constant, $\frac{C_t}{L_t}$, $\frac{Y_t}{L_t}$ and $\frac{K_t}{L_t}$ grow as the same rate as the growth rate of technology, g.

Homework: Review the property of the Solow model.

Let me estimate the steady state value of GDP per capita. Assume that the production function is Cobb-Douglas, $f(k) = k^\alpha$. The steady state conditions (28) and (29) are rewritten as

$$\begin{aligned} \alpha \left(k_t^*\right)^{\alpha-1} &= \rho + \delta + n + \theta g, \qquad (31) \\ c_t^* &= \left(k_t^*\right)^\alpha - (g + n + \delta) k_t^*. \qquad (32) \end{aligned}$$

Equation (31) implies that

$$k_t^* = \left[\frac{\alpha}{\rho + \delta + n + \theta g}\right]^{\frac{1}{1-\alpha}}$$

Therefore

$$\frac{Y_t}{L_t} = \left[\frac{\alpha}{\rho + \delta + n + \theta g}\right]^{\frac{\alpha}{1-\alpha}} T_0 e^{gt}$$

Similar to the Solow model, the technology level has a positive impact on GDP per capita, and population growth has negative impact. Since the saving rate is endogenized, the saving rate is replaced by the parameters on the utility function, ρ and θ. The more the agent discount the future, (large ρ), the lower the steady state value of per capita GDP. Similarly, the larger the marginal rate of substitution $\frac{1}{\theta}$, the larger per capita GDP is. In order to understand the reason, let me derive the steady state value of the saving rate.

$$\begin{aligned} s &= \frac{\left(k_t^*\right)^\alpha - c_t^*}{\left(k_t^*\right)^\alpha}, \\ &= \frac{\left(k_t^*\right)^\alpha - \left[\left(k_t^*\right)^\alpha - (g + n + \delta) k_t^*\right]}{\left(k_t^*\right)^\alpha}, \\ &= (g + n + \delta) \left(k_t^*\right)^{1-\alpha}, \\ &= \frac{\alpha (g + n + \delta)}{\rho + \delta + n + \theta g}. \end{aligned}$$

When ρ is large, the agent largely discounts his future. Since today is more important than tomorrow, the agent saves less. Therefore, it lowers per capita GDP. When θ is small, the marginal rate of substitution is large. The agent is willing to change his consumption in response to the the change in the return. When the economy is growing, the return is high. Therefore, this behavior implies that the agent saves more. Hence, small θ induces high GDP per capita.

Homework: Derive the consumption per capita on the steady state.

Market Economy: So far, I discussed a command economy: the social planner maximizes the representative consumer's utility function subject to the resource constraint. But how does this command economy relate to the market economy? In order to answer this question, remember that the first and second welfare theorem. The first welfare theorem says that the market economy is Pareto optimal. The second welfare theorem says that the Pareto optimum allocation can be replicated by a market economy with the income transfer. Note that our command economy is Pareto optimum and since every agent is identical, there is no reason to talk income transfer in our model. Hence, there must have prices which support our equilibrium.

Homework: Review the first and second welfare theorem.
Homework: Prove that our economy is Pareto optimum.

Define the rental price, r_t, the interest rate, i_t and the wage rate, w_t, as follows:

$$\begin{aligned} r_t &= f'(k_t) && (33) \\ i_t &= r_t - \delta && (34) \\ w_t &= \left[f(k_t) - f'(k_t)\,k_t\right] T_t && (35) \end{aligned}$$

Note that equations (33) and (35) are the necessary and sufficient conditions of the following profit maximization problem:

$$\max_{K_t, L_t} \left\{F(K_t, T_t L_t) - r_t K_t - w_t L_t\right\},$$

where $f(k_t) = F\left(\frac{K_t}{T_t L_t}, 1\right)$. Similarly, equation (34) is the arbitrage conditions at the financial market. It is derived from equation (33) and (35) that

$$f(k_t) = r_t k_t + w_t^*. \tag{36}$$

where $w_t^* = \frac{w_t}{T_t}$. The variable, w_t^* can be seen as the wage rate per efficiency unit. The feasibility condition implies that $K_t = A_t$ and $L_t = N_t$, where A_t and N_t are available assets and labor at date t. Define $a_t = \frac{A_t}{T_t N_t}$. Using equations (33), (34) and (36), the first order conditions for the social planner problem will be rewritten as

$$\begin{aligned}
\lambda_t &= U'(c_t T_t) T_t && (37)\\
\dot{\lambda}_t &= \rho\lambda_t - \lambda_t [i_t - (g+n)],\ 0 = \lim_{t\to\infty} \lambda_t a_t e^{-\rho t} && (38)\\
\dot{\mu}_t &= \rho\mu_t - U'(c_t T_t) c_t - \mu_t g,\ 0 = \lim_{t\to\infty} \mu_t T_t e^{-\rho t} && (39)\\
\dot{a}_t &= [i_t - (g+n)] a_t + w_t^* - c_t,\ a_0 \ is\ given && (40)\\
\dot{T}_t &= gT_t,\ T_0 \ is\ given && (41)
\end{aligned}$$

Notice that these equation can be seen as the first order conditions of the following consumer's maximization problem.

$$\begin{aligned}
&\max_{c_t} \int_0^\infty e^{-\rho t} U(c_t T_t)\, dt \\
s.t.\ \dot{a}_t &= [i_t - (g+n)] a_t + w_t^* - c_t,\ a_0 \ is\ given \\
\dot{T}_t &= gT_t,\ T_0 \ is\ given
\end{aligned}$$

Since the first order conditions for the social planner problem are sufficient for the social planner's problem, and the first order conditions for the consumer's maximization problem is sufficient for the consumer's maximization problem, two problems are equivalent. In other words, the allocation pattern shown by the social planner problem are reproduced by the market equilibrium.

Homework: Show that the above consumer's maximization problem is rewritten as

$$\begin{aligned}
&\max_{c_t} \int_0^\infty e^{-\rho t} U\left(\frac{C_t}{N_t}\right) dt \\
s.t.\ \dot{A}_t &= i_t A_t + w_t T_t N_t - C_t,\ A_0 \ is\ given \\
\dot{T}_t &= gT_t,\ T_0 \ is\ given \\
\dot{N}_t &= nN_t,\ N_0 \ is\ given.
\end{aligned}$$

Consumer: Let me investigate the first order conditions of the representative consumer. Assume the constant relative risk aversion utility function:

$$\begin{aligned}
U(c_t T_t) &= \frac{(c_t T_t)^{(1-\theta)} - 1}{1-\theta},\ if\ \theta \neq 1, \\
&= \log(c_t T_t),\ if\ \theta = 1.
\end{aligned}$$

I can derive the Euler equation from equations (37) and (38):

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}[i_t - (\rho + n + \theta g)]. \tag{42}$$

Homework: Derive the Euler equation.

Since $c_t = \frac{C_t}{T_t N_t}$, the Euler equation can be modified:

$$\frac{\frac{d\left(\frac{C_t}{N_t}\right)}{dt}}{\left(\frac{C_t}{N_t}\right)} = \frac{1}{\theta}\left[i_t - (\rho + n)\right]$$

Note that the Euler equation implies that the agent raises consumption as long as the interest rate i_t is larger than the sum of discount rate and population growth. It is easier to interpret this condition when the population growth rate is 0. When there is no population growth, the agent compares between the interest rate and the discount rate. When the interest rate is higher than the discount rate, the return from saving is larger then the disutility from deferring consumption. Therefore, the agent saves more and consumes more in the future. Therefore, consumption will grow.

I would like to show that consumption depends on permanent income. To do so, we need to solve differential equation. Firstly, I would like to show the solution for typical differential equations. Then I apply these method to the current problem. The next lemma shows the solution when the initial condition is given.

Lemma 2 *Suppose that x_t follows a differential equation:*

$$\dot{x}_t = a_t + b_t x_t, \ x_\tau \text{ is given}$$

Then the solution for this differential equation is

$$x_T = e^{\int_\tau^T b_s ds}\left[x_\tau + \int_\tau^T a_t e^{-\int_\tau^t b_s ds} dt\right]$$

Proof. *Note that the differential equation can be rewritten as*

$$\begin{aligned} a_t e^{-\int_\tau^t b_s ds} &= \dot{x}_t e^{-\int_\tau^t b_s ds} - b_t x_t e^{-\int_\tau^t b_s ds} \\ &= \frac{d\left[x_t e^{-\int_\tau^t b_s ds}\right]}{dt}. \end{aligned}$$

Integrate the both side of the above equation from τ to T:

$$\begin{aligned} \int_\tau^T a_t e^{-\int_\tau^t b_s ds} dt &= \int_\tau^T \frac{d\left[x_t e^{-\int_\tau^t b_s ds}\right]}{dt} dt \\ &= x_T e^{-\int_\tau^T b_s ds} - x_\tau \end{aligned}$$

The desired result is immediate from the last equation. ∎

The next lemma derives the solution with other boundary condition.

Lemma 3 *Suppose that x_t follows a differential equation:*

$$\begin{aligned} \dot{x}_t &= a_t + b_t x_t, \\ 0 &= \lim_{T\to\infty} x_T e^{-\int_t^T b_s ds} \end{aligned}$$

Then the solution for this differential equation is

$$x_\tau = -\int_\tau^\infty a_t e^{-\int_\tau^t b_s ds} dt$$

Proof. *Using the proof of the previous lemma, we can derive*

$$\int_\tau^T a_t e^{-\int_\tau^t b_s ds} dt = x_T e^{-\int_\tau^T b_s ds} - x_\tau$$

Since $0 = \lim_{T\to\infty} x_T e^{-\int_t^T b_s ds}$, *the desired result is immediate.* ■

Let me apply the above methods to equation (38).

$$\begin{aligned} 0 &= \dot{\lambda}_t - \{\rho - [i_t - (g+n)]\} \lambda_t \\ &= \dot{\lambda}_t e^{-\int_0^t \{\rho - [i_s - (g+n)]\} ds} - \{\rho - [i_t - (g+n)]\} \lambda_t e^{-\int_0^t \{\rho - [i_s - (g+n)]\} ds} \\ &= \frac{d\left[\lambda_t e^{-\int_0^t \{\rho - [i_s - (g+n)]\} ds}\right]}{dt} \end{aligned}$$

Hence,

$$\begin{aligned} 0 &= \int_0^T \frac{d\left[\lambda_t e^{-\int_0^t \{\rho - [i_s - (g+n)]\} ds}\right]}{dt} dt \\ &= \lambda_T e^{-\int_0^T \{\rho - [i_s - (g+n)]\} ds} - \lambda_0 \end{aligned}$$

This result implies that

$$\lambda_T = \lambda_0 e^{\int_0^T \{\rho - [i_s - (g+n)]\} ds}$$

Substitute this result into the Transversality condition of equation (38)

$$\begin{aligned} 0 &= \lim_{T\to\infty} \lambda_T a_T e^{-\rho T} \\ &= \lim_{T\to\infty} \lambda_0 e^{\int_0^T \{\rho - [i_s - (g+n)]\} ds} a_T e^{-\rho T} \end{aligned}$$

Hence we can derive the following condition:

$$0 = \lim_{T\to\infty} a_T e^{-\int_0^T [i_s - (g+n)] ds}. \tag{43}$$

This condition is called no Ponzi game condition. As I have shown, when asset, a_t, is always positive, we can derive no Ponzi game condition from the transversality condition of the original problem. Hence, it can be seen as one of the necessary and sufficient conditions. If we allow to have debt, a_t could be negative. But if a_t can be negative, it is known that we need a different transversality condition for optimal conditions:

$$\lim_{T\to\infty} \lambda_T e^{-\rho T} = 0.$$

In this case, we can not derive equation (43). Therefore, no Ponzi game condition is not a part of optimal condition. Many economists simply assume equation (43) as one of the regularity condition. The economic interpretation of no Ponzi game condition is that debt can not increase faster than the interest rate. Otherwise the debt can not be repaid forever.

Using no Ponzi game condition, the flow budget constant (40) can is equivalent to an intertemporal budget constraint:

$$\begin{aligned} \int_0^\infty c_t e^{-\int_0^t [i_s-(g+n)]ds} dt &= h_0 + a_0 \qquad (44) \\ where\ h_0 &= \int_0^\infty w_t^* e^{-\int_0^t [i_s-(g+n)]ds} dt \end{aligned}$$

Homework: Derive the intertemporal budget constraint (44). Hint: derive the following equation from the flow budget constraint (40) using the previous lemma.

$$\int_0^T c_t e^{-\int_0^t [i_s-(g+n)]ds} dt = \int_0^T w_t^* e^{-\int_0^t [i_s-(g+n)]ds} dt + a_0 - a_T e^{-\int_0^T [i_s-(g+n)]ds}$$

The intertemporal budget constraint implies that the present value of consumption flow is equal to the total wealth, which is the sum of nonhuman wealth, a_0, and of human wealth, h_0. It also shows that human wealth, h_0, is the present value of labor income.

Euler equation (42) determines the rate of change in consumption. Solving the Euler equation gives us information about the level of consumption. Applying the previous lemma to the Euler equation, we can derive

$$c_t = c_0 e^{\int_0^t \frac{1}{\theta}[i_t-(\rho+n+\theta g)]ds} \qquad (45)$$

Homework: Derive equation (45).

Substitute equation (45) into the intertemporal budget constraint (44). We can derive the following consumption function.

$$\begin{aligned} c_0 &= \beta_0 \left(h_0 + a_0\right) \qquad (46) \\ where\ \beta_0 &= \left[\int_0^{\infty} e^{\int_0^t \frac{(1-\theta)[i_t - n] - \rho}{\theta} ds} dt\right]^{-1} \end{aligned}$$

Homework: Make sure that equation (46) is correct.

Equation (46) says that current consumption is a linear function of the permanent income, which is the sum of human and nonhuman wealth. The parameter β_0 can be interpreted as the propensity to consume out of wealth. As you can see, the propensity to consume depends on the expected path of the interest rate. Note that this is consistent with the permanent income hypothesis, which was discussed in Modern Economics 1.

Government in the market economy: Let me introduce government in our market economy[1]. Firstly, I discuss the balanced budget changes in government spending. Secondly, I extend our argument with government debt. Lastly, I discuss the case that government poses distortionary taxation.

Balanced Budget: Let me first consider the case government expenditure is always balanced by tax revenue. I show that an increase in government expenditure crowds out private consumption. This result is corresponding to the impact of stabilization policy in the long run in Modern Macroeconomics 1. I also analyze the temporal change in government policy, which we can not discuss by the static model in Modern Macroeconomics 1.

Assume that government spends G_t every period and finance it by a lump sum tax, τ_t. Assume that government's budget constraint must be satisfied every period: $G_t = \tau_t$. Then the representative consumer with the constant relative risk aversion

[1]Strictly speaking, introducing government causes technical problems. Remember that we proved the equivalence of market economy and command economy using the first and second welfare theorem. Since we can show the existence of command economy, the first and second welfare theorem ensure the existence of the market economy. When I introduce government, the first and second welfare theorem may not hold. If not, we can not ensure the existence of the market economy. Since this is a technical issue, I do not discuss here.

utility function solves the following problem:

$$
\begin{aligned}
& \max_{c_t} \int_0^{\infty} e^{-\rho t} \frac{\left(\frac{C_t}{N_t}\right)^{(1-\theta)} - 1}{1-\theta} dt \\
s.t.\ \dot{A}_t &= i_t A_t + w_t T_t N_t - C_t - G_t,\ A_0 \textit{ is given} \\
\dot{T}_t &= g T_t,\ T_0 \textit{ is given} \\
\dot{N}_t &= n N_t,\ N_0 \textit{ is given.}
\end{aligned}
$$

Define government expenditure per an efficiency unit, $\gamma_t = \frac{G_t}{T_t N_t}$. Then the problem is equivalent to

$$
\begin{aligned}
& \max_{c_t} \int_0^{\infty} e^{-[\rho-(1-\theta)g]t} \frac{(c_t)^{(1-\theta)} - 1}{1-\theta} dt \\
s.t.\ \dot{a}_t &= [i_t - (g+n)]\, a_t + w_t^* - c_t - \gamma_t,\ a_0 \textit{ is given}
\end{aligned}
$$

Homework: Check the equivalence.

The Euler equation of this problem is

$$
\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}\left[i_t - (\rho + n + \theta g)\right],
$$

and the transversality condition is

$$
0 = \lim_{T\to\infty} c_t^{-\theta} a_t e^{-[\rho-(1-\theta)g]t}.
$$

Homework: Derive the Euler equation and the transversality condition.

Using the capital market clearing condition, $K_t = A_t$, the labor market clearing condition, $L_t = N_t$ and the firm's maximization conditions: (33), (34) and (35), economy will be summarized by the following two equations:

$$
\begin{aligned}
\frac{\dot{c}_t}{c_t} &= \frac{1}{\theta}\left[f'(k_t) - (\rho + \delta + n + \theta g)\right], \\
\dot{k}_t &= f(k_t) - c_t - \gamma_t - (g+n+\delta)\, k_t, \\
0 &= \lim_{t\to\infty} (c_t)^{-\theta} k_t e^{-[\rho-(1-\theta)g]t},\ k_0 \textit{ is given.}
\end{aligned}
$$

The main difference between these conditions and the previous social planner problem is that the transition function depends on the stream of γ_t. Assume that $\gamma_t = \gamma$.

That is, government expenditure per capita expands the same rate as productivity growth. Then we can derive Phase diagram as before.

FIGURE 2 MUST BE HERE

Look at Figure 2. As you can see, the steady state value of consumption goes down, while the steady state value of capital stock does not change. That is, the impact of an increase in government expenditure crowds out private consumption. Note that this is corresponding to stabilization policy in the long run in Modern Macroeconomics 1.

We can analyze the impact of a temporal policy change in this framework, which we could not do in static stabilization. Assume that initially the economy locates on the steady state without government expenditure and that government decides to spend γ at date t_0 without any announce. Assume also that everybody knows that government will stop spending at date t_1. When government decides to spends γ, this is surprise. The agent suddenly drops his consumption as before. But he knows that government will end it at date t_1. It means that he must optimally behave at date t_1. Since his optimal behavior is determined by the Euler equation, it is not optimal for consumption to jump. Formally speaking, between t_0 and t_1, the economy must follow

$$\begin{aligned}\frac{\dot{c}_t}{c_t} &= \frac{1}{\theta}\left[f'(k_t) - (\rho + \delta + n + \theta g)\right], \\ \dot{k}_t &= f(k_t) - c_t - \gamma - (g + n + \delta) k_t,\end{aligned}$$

and after t_1, economy must follow

$$\begin{aligned}\frac{\dot{c}_t}{c_t} &= \frac{1}{\theta}\left[f'(k_t) - (\rho + \delta + n + \theta g)\right], \\ \dot{k}_t &= f(k_t) - c_t - (g + n + \delta) k_t,\end{aligned}$$

and at t_1 economy must switch the system without any discontinuity. This possibility is depicted by Figure 3.

FIGURE 3 MUST BE HERE

At t_0, the agent drops his consumption, but he does not drop it as much as he would do when the permanent change occurs. Then Figure 3 implies that capital eventually decreases and consumption eventually increases. Since the agent knows that the system would return to the original at date t_1, he must reach the saddle path of the original system at date t_1. If he succeeds to do so, he can be on the optimal path without any jump at date t_1. Anticipating this result, the agent optimally chooses the magnitude of jump.

Figure 3 shows that a temporal increase in government expenditure initially forces consumption drops. But eventually consumption goes back to original level. Government expenditure not only crowds out private consumption, but also private investment. Therefore, it lowers capital stock initially. When government stop spending, it ends the crowding out effect on private investment, and the agents start accumulate capital stock. Eventually, economy will return to original.

Recardian Equivalence: Since I analyze stabilization policy in a static model, I can not discuss the impact of debt financing in an equilibrium model in Modern Macroeconomics 1. Let me put government debt in this model. I want to show Recardian Equivalence, which was discussed in Modern Macroeconomics 1.

Let B_t denote government debt at date t. The accumulation of government debt follows

$$\dot{B}_t = i_t B_t + G_t - \tau_t.$$

Assume no Ponzi game condition for government budget constraint.

$$0 = \lim_{t\to\infty} B_t e^{-\int_0^t i_s ds}.$$

This constraint implies that government debt can not explore faster than interest rate. If this condition is violated, government cannot repay the debt. As usual, define debt per unite of effective labor as $b_t = \frac{B_t}{T_t N_t}$ and taxes per unit of effective labor $\tau_t^* = \frac{\tau_t}{T_t N_t}$. Then

$$\dot{b}_t = \gamma_t - \tau_t^* + [i_t - (n+g)]\, b_t$$

and

$$0 = \lim_{t\to\infty} b_t e^{-\int_0^t [i_s - (n+g)] ds}$$

Using these two conditions, I can derive the following government intertemporal budget constraint:

$$b_0 + \int_0^\infty \gamma_t e^{-\int_0^t [i_s - (n+g)] ds} dt = \int_0^\infty \tau_t^* e^{-\int_0^t [i_s - (n+g)] ds} dt. \tag{47}$$

Homework: Derive the above three equations.

The government intertemporal budget constraint (47) implies that current debt plus the present value of the stream of the future government expenditure must be financed by the present value of the stream of tax revenue.

Given government debt, the dynamic budget constraint of a representative consumer is

$$\dot{a}_t = [i_t - (g+n)]\, a_t + w_t^* - c_t - \tau_t^*$$

Assuming no Ponzi game condition (43), we can derive an intertemporal budget constraint with taxes:

$$\int_0^\infty c_t e^{-\int_0^t [i_s - (g+n)] ds} dt = h_0 + a_0 - \int_0^\infty \tau_t^* e^{-\int_0^t [i_s - (g+n)] ds} dt \tag{48}$$

Because the tax burden lowers the consumer's income, the present value of the stream of tax revenue must be subtracted by the permanent income.

Homework: Derive the intertemporal budget constraint with taxes.

Since the agent can invest either in physical capital or government debt, capital market clearing condition implies

$$a_t = k_t + b_t. \tag{49}$$

Note that since market is competitive, the return to physical capital and government debt must be the same, i_t, in this model.

Substituting the government intertemporal budget constraint (47) and the capital market clearing condition (49) into the individual budget constraint (48),

$$\int_0^\infty c_t e^{-\int_0^t [i_s-(g+n)]ds} dt = h_0 + k_0 - \int_0^\infty \gamma_t e^{-\int_0^t [i_s-(n+g)]ds} dt \tag{50}$$

Note that the new budget constraint (50) does not include neither debt, b_t, and taxes, τ_t^*. Therefore, the method of financing does not change the budget constraint of the representative consumer. Note also that since taxation is lump sum, it does not change consumer behavior. Hence, for a given path of government expenditure, the method of financing, through lump-sum taxation or debt financing does not change the consumer behavior.

Distortionary Taxation: The benefit of explicitly solving the consumer behavior is that we can analyze the impact of policy change on the saving rate. Here, I discuss the impact of capital tax on our economy. Let τ_k denote constant capital income tax rate. In order to focus the impact on capital income tax, I simply assume that government distributes tax revenue to every agent. Let z_t denote that lump-sum transfer per unit of effective labor. Then the representative consumer with constant relative risk aversion solves

$$\max_{c_t} \int_0^\infty e^{-[\rho-(1-\theta)g]t} \frac{(c_t)^{(1-\theta)}-1}{1-\theta} dt$$

$$s.t.\ \dot{a}_t = [(1-\tau_k)\, i_t - (g+n)]\, a_t + w_t^* + z_t - c_t,\ a_0 \ is\ given$$

Note that because of capital income tax, the agent's capital income is $(1-\tau_k)\, i_t a_t$. Euler equation of this problem is

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}\left[(1-\tau_k)\, i_t - (\rho+n+\theta g)\right],$$

and the transversality condition is

$$0 = \lim_{T\to\infty} c_t^{-\theta} a_t e^{-[\rho-(1-\theta)g]t}.$$

Homework: Derive the above two equations.

Since government's budget constraint is balanced at each date,

$$\tau_k i_t a_t = z_t.$$

Substituting this equation to the agent's budget condition,

$$\dot{a}_t = [i_t - (g+n)]\, a_t + w_t^* - c_t,$$

Using the capital market clearing condition, $K_t = A_t$, the labor market clearing condition, $L_t = N_t$ and the firm's maximization conditions: (33), (34) and (35), economy is summarized by the following two equations:

$$\begin{aligned}
\frac{\dot{c}_t}{c_t} &= \frac{1}{\theta}\left\{(1-\tau_k)\left[f'(k_t) - \delta\right] - (\rho + n + \theta g)\right\}, \\
\dot{k}_t &= f(k_t) - c_t - (g+n+\delta)\, k_t, \\
0 &= \lim_{t\to\infty} (c_t)^{-\theta}\, k_t e^{-[\rho-(1-\theta)g]t},\ k_0 \textit{ is given.}
\end{aligned}$$

Homework: Derive the above two equations.

Note that the transition equation of physical capital is the same as the social planner case; the Euler equation differs. On the steady state

$$f'(k_t^*) = \delta + \frac{(\rho + n + \theta g)}{(1-\tau_k)} \tag{51}$$

$$c_t^* = f(k_t^*) - (g+n+\delta)\, k_t^* \tag{52}$$

Equation (51) shows that an increase in τ_k increases $f'(k_t^*)$. Therefore it lowers capital stock on the steady state. Since capital income tax lowers the marginal benefit of saving, the agent consumes more than optimal. Therefore it lowers the steady state level of capital stock. Look at Figure 4. Figure 4 shows that since the steady state level of capital stock is lower than that in the social planner model, the steady state level of consumption is also lower.

FIGURE 4 MUST BE HERE

Let me estimate the steady state level of per capita income. Assume $f(k) = k^\alpha$. Then it is derived from equation (51) that

$$\alpha (k_t^*)^{\alpha-1} = \left[\frac{\delta(1-\tau_k) + (\rho+n+\theta g)}{(1-\tau_k)}\right]$$

Hence

$$k_t^* = \left[\frac{\alpha(1-\tau_k)}{\delta(1-\tau_k)+(\rho+n+\theta g)}\right]^{\frac{1}{1-\alpha}}$$

Hence

$$\frac{Y_t}{N_t} = \left[\frac{\alpha(1-\tau_k)}{\delta(1-\tau_k)+(\rho+n+\theta g)}\right]^{\frac{\alpha}{1-\alpha}} T_t$$

You can see that an increase in τ_k lowers per capita in GDP. To see the reason, let me derive the saving rate on the steady state.

$$s = \frac{\alpha(1-\tau_k)(g+n+\delta)}{\delta(1-\tau_k)+(\rho+n+\theta g)}.$$

As you can see, the capital income tax lowers the saving rate. Therefore it lowers capital stock per capita and GDP per capita on the steady state.

Homework: Derive the saving rate on the steady state.

4 Overlapping Generation Model

There is another popular model to analyze economic growth. It is called an overlapping generation model (OGM). The main difference from the representative agent model is that there is turnover in the population. New generation comes to society; old generation leaves. The main purpose of this section is to show that (1) when we have turnover in the population, economy may not be Pareto optimal and (2) a representative agent model can be viewed as the model in which parents care about their children.

Basic Model: Let me describe the basic framework of the OGM. For the simplest case, each agent lives in two periods; young and old. In each period, new young enters the economy and old leaves. Therefore there is turnover in the population in each period. The agents are identical in their generation, but differs across generation. Let me first describe the behavior of consumers in this economy.

Consumers: The agent with a constant risk aversion solves the following problem:

$$\max_{C_{yt},C_{ot+1}} \left\{\frac{C_{yt}^{(1-\theta)}-1}{1-\theta} + \frac{1}{1+\rho}\frac{C_{ot}^{(1-\theta)}-1}{1-\theta}\right\}$$

$$s.t. S_t = w_t - C_{yt} \quad (53)$$

$$C_{ot} = (1+i_{t+1}) S_t \quad (54)$$

where C_{yt} and C_{ot} are consumption when he is young and old, respectively. When the agent is young, he earns wage w_t. He decides whether he consumes today or save it for the next period. When he gets age, he retires and receives income from his saving. Since he does not have the next period, he consumes all his wealth when he is old.

This model can be simplified by

$$\max_{S_t}\left\{\frac{[w_t - S_t]^{(1-\theta)}-1}{1-\theta}+\frac{1}{1+\rho}\frac{[(1+i_{t+1})\,S_t]^{(1-\theta)}-1}{1-\theta}\right\}$$

The first order condition is

$$[C_{yt}]^{-\theta}=\frac{(1+i_{t+1})\,[C_{ot+1}]^{-\theta}}{1+\rho}$$

Hence,

$$\frac{C_{ot+1}}{C_{yt}}=\left(\frac{1+i_{t+1}}{1+\rho}\right)^{\frac{1}{\theta}} \tag{55}$$

This is OGM version of the Euler equation. To see this, if we assume that $C_{ot} = C_{yt} = C_t$,

$$\begin{aligned}\frac{\dot{C}_t}{C_t} &\approx \log C_{t+1}-\log C_t\\ &= \frac{1}{\theta}\left[\log(1+i_t)-\log(1+\rho)\right]\\ &\approx \frac{1}{\theta}\left[i_t-\rho\right]\end{aligned}$$

This is the same as the Euler equation of the representative agent model without population growth. The agent compares the interest rate and the discount rate. When the interest rate is higher than the discount rate, he saves more and increases his consumption tomorrow. When the discount rate is larger, he increases his consumption today.

Combing two budget constraints (53) and (54), I can derive the intertemporal budget constraint:

$$C_{yt}+\frac{C_{ot+1}}{1+i_{t+1}}=w_t. \tag{56}$$

It shows that the present value of the stream of lifetime consumption is equal to the present value of lifetime income, which, in this case, wages during young. This is similar to the intertempral budget constraint in the representative agent model. Substituting the Euler equation (55) into (56),

$$\left[1+\frac{\left(\frac{1+i_{t+1}}{1+\rho}\right)^{\frac{1}{\theta}}}{1+i_{t+1}}\right]C_{yt}=w_t.$$

Therefore, consumption is linear in wages.

$$C_{yt} = \frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+i_{t+1})^{\frac{1-\theta}{\theta}}} w_t$$

We can derive the saving rate, $s(i_t)$, as the function of the interest rate:

$$\begin{aligned} s(i_{t+1}) &= \frac{C_{yt} - w_t}{w_t} \\ &= \frac{(1+i_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+i_{t+1})^{\frac{1-\theta}{\theta}}} \end{aligned}$$

As you can see, the saving rate can be an increasing or decreasing function of the interest rate:

$$\begin{aligned} s'(i_{t+1}) &> 0,\ if\ 1 > \theta \\ s'(i_{t+1}) &< 0,\ if\ 1 < \theta \\ s'(i_{t+1}),\ if\ 1 &= \theta \end{aligned}$$

This is the result of a usual substitution effect and income effect. When the interest rate increases, the return to saving increases. Therefore the agent saves more (substitution effect). On the other hand an increase in the interest rate implies an increases in lifetime income. Therefore, the agent can consumes more. As I discussed in the representative agent model, the parameter $\frac{1}{\theta}$ can be interpreted as the elasticity of substitution between consumption today and tomorrow. It measures the sensitivity of consumption to the change in price. When $\frac{1}{\theta}$ is larger than 1, θ is smaller than 1, the agent is sensitive to the change in the return to saving. Therefore, substitution effect dominates income effect. The saving rate is an increasing function of the interest rate. Using this saving rate, the gross saving is

$$S_t = s(i_{t+1})\, w_t$$

Firm: Firms' maximization conditions are the same as before. The following profit maximization condition characterizes the firms' behavior:

$$\begin{aligned} r_t &= f'(k_t) \\ w_t &= [f(k_t) - f'(k_t)\, k_t]\, T_t \end{aligned}$$

As usual, I express the first order conditions by the efficiency unit term.

Intermediation: The following arbitrage condition is also the same as before:

$$i_t = r_t - \delta$$

Market Clearing Conditions: Since only the young works in this economy, the demand for labor must be equal to the population of the young.

$$L_t = N_{yt}$$

where N_{yt} is the population of the young. On the other hand, since when the agent becomes old, he consumes every asset he has. Therefore, the next period asset supply is equal to saving which the current young make during this period. Since the demand for capital must be equal to the supply of asset,

$$K_{t+1} = S_t N_{yt}.$$

We assume that the growth rate of productivity and population is g and n, respectively:

$$\begin{aligned} T_{t+1} &= (1+g) T_t \\ N_{yt+1} &= (1+n) N_{yt} \end{aligned}$$

Using efficiency unit, both market clearing condition is summarized by

$$k_{t+1} (1+g)(1+n) T_t = S_t.$$

Note that the denominator of k_{t+1} is not total population, but the number of workers. Since every agent works in the representative agent model, we do not need distinguish between the number of workers and population. But we must clearly distinguish in this model.

Homework: Make sure that you can derive equation (61).

Equilibrium: Let me define the market equilibrium.

Definition 4 *The market equilibrium consists of the sequence of* $\{(S_t, i_{t+1}, w_t, r_t, k_t)\}_{t=0}^{\infty}$, *which satisfies*

1. *Consumer maximizes his utility:*

$$\begin{aligned} S_t &= s(i_{t+1}) w_t && (57) \\ \textit{where } s(i_{t+1}) &= \frac{(1+i_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+i_{t+1})^{\frac{1-\theta}{\theta}}} \end{aligned}$$

2. *Firm maximizes its profits:*

$$\begin{aligned} r_t &= f'(k_t) && (58) \\ w_t &= [f(k_t) - f'(k_t) k_t] T_t && (59) \end{aligned}$$

3. *The arbitrage condition:*

$$i_{t+1} = r_{t+1} - \delta \tag{60}$$

4. *Market clearing condition:*

$$k_{t+1}(1+g)(1+n)T_t = S_t. \tag{61}$$

Combining equilibrium conditions, the following equation summarizes the dynamics of our economy:

$$k_{t+1} = \frac{\left(f'(k_{t+1}) + 1 - \delta\right)^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + \left(f'(k_{t+1}) + 1 - \delta\right)^{\frac{1-\theta}{\theta}}} \frac{\left[f(k_t) - f'(k_t)k_t\right]}{(1+g)(1+n)} \tag{62}$$

Homework: Derive equation (62).

As you can see, this is a nonlinear first order difference equation. Hence, potentially, we can solve it given an initial condition k_0. Unfortunately, we can not say much about the property of this dynamic equation. It is well known that the OGM can produce the variety of dynamics. It is possible to have multiple steady states in the OGM. In another case, we can not determine the dynamics of OGM. Moreover, OGM can yield a chaotic fluctuation. The following figures are some examples.

FIGURE MUST BE HERE.

The dynamics of the OGM depends on the parameters of the production function and the utility function. This can be seen as the merit and demerit of OGM. On one hand, it gives us possible explanations for several puzzling evidence including business cycle. On the other hand, it is difficult to tell where we stand and what would be policy implication. Anything goes.

Two additional assumptions simplify the dynamics of our solution. Assume that $\theta = 1$ and the production function is Cobb-Douglas $f(k) = k^{\alpha}$.. Then equation (62) is

$$k_{t+1} = \frac{(1-\alpha)(k_t)^{\alpha}}{(2+\rho)(1+g)(1+n)}.$$

Look at Figure **?**. It shows that there is a unique steady state and economy globally and monotonically converges to the steady state. Let me derive the steady state value of k_t. On the steady state $k_t = k_{t+1} = k^*$. Then

$$k^* = \left[\frac{1-\alpha}{(2+\rho)(1+g)(1+n)}\right]^{\frac{1}{1-\alpha}} \tag{63}$$

Dynamic Inefficiency: Let me discuss the potential inefficiency of the OGM. To do so, I examine the level of capital stock which maximizes consumption per capita on the steady state, which is called the golden rule level of capital stock. Then I compare this capital stock and the steady state capital stock which the OGM can reach. Note that if the steady state capital stock level is larger than the golden rule level of capital stock, we can easily find new allocation in which everybody is better off. To see this, lower the capital stock on the steady state to the golden rule level during period t. Since the agent can enjoy more consumption during period t, the agent is better off. From the next period, economy reaches the golden level of capital stock. Since the golden rule level of capital stock maximizes consumption per capita on the steady state, the following generations can enjoy higher consumption and they are better off. In other word, the steady state is not efficient. This is called the dynamic inefficiency.

Firstly, note that maximizing consumption per capita is the same as maximizing consumption per workers:

$$\begin{aligned} \frac{C_t}{N_{yt}+N_{ot}} &= \frac{C_t}{N_{yt}+\frac{N_{yt}}{1+n}}, \\ &= \frac{1+n}{2+n}\frac{C_t}{N_{yt}}, \end{aligned}$$

where $C_t = C_{yt}N_{yt} + C_{ot}N_{ot}$. Hence, I find the level of capital stock which maximizes consumption per workers given any level of technology. Social planner faces resource constraint any time. The resource constraint is

$$K_{t+1} = F\left(K_t, T_t N_{yt}\right) - C_t + (1-\delta)K_t$$

Using efficiency unit,

$$k_{t+1}(1+g)(1+n) = f(k_t) - c_t + (1-\delta)k_t$$

where $c_t = \frac{C_t}{T_t N_{yt}}$ and $k_t = \frac{K_t}{T_t N_{yt}}$. On the steady state $k_t = k_{t+1} = k^{**}$ and $c_t = c^{**}$. Hence,

$$c^{**} = f\left(k^{**}\right) - (\delta+g+n+ng)\,k^{**}$$

Hence, the level of capital stock which maximizes consumption per workers given the level of productivity is

$$\frac{dc^{**}}{dk^{**}} = f'\left(k^{GR}\right) - (\delta+g+n+ng) = 0$$

Note that if $ng \approx 0$, this golden rule is the same as the golden rule in the representative agent model. Assume that $f(k) = k^{\alpha}$, the golden rule level of capital stock per unit of effective labor is

$$k^{GR} = \left[\frac{\alpha}{\delta+g+n+ng}\right]^{\frac{1}{1-\alpha}} \tag{64}$$

Compare equation (63) and (64), I get

$$k^{GR} < k^{*},\ if\ \frac{\alpha}{\delta+g+n+ng} < \frac{1-\alpha}{(2+\rho)(1+g)(1+n)}.$$

$$
\begin{aligned}
\frac{\alpha}{1-\alpha} &< \frac{\delta+g+n+ng}{(2+\rho)(1+g)(1+n)} \\
&= \frac{(1+g)(1+n)-(1-\delta)}{(2+\rho)(1+g)(1+n)} \\
&= \frac{1}{(2+\rho)}\left[1-\frac{(1-\delta)}{(1+g)(1+n)}\right]
\end{aligned}
$$

Hence, there exists parameter values with which the steady state is not Pareto optimal. This result is contrasted with the one of the representative agent model in which the allocation is Pareto optimal. This result sounds surprising since market is competitive and no externality in Overlapping generation model. The main reason is that we have the infinite number of agents in our economy. The social planner can transfer resources from young to old without a market. Obviously, the current old is better off and the current young is worse off. However, the social planner can compensate the current young when he becomes old by transferring resources from the next generation. Since we have the infinite number of generations, nobody may be worse off. This is the reason for inefficiency. This inefficiency is called the dynamic inefficiency.

Altruism: I would like to modify the basic model. Suppose that the agents cares about their children. More concretely, the young during period t maximizes the following utility function:

$$
U_t = \frac{C_{yt}^{1-\theta}-1}{1-\theta} + \frac{1}{1+\rho}\left[\frac{C_{ot+1}^{1-\theta}-1}{1-\theta} + \frac{1+n}{1+\psi}U_{t+1}\right],
$$

where ψ is the measure of selfishness. If $\psi = 0$, parents treat their children like themselves. The variable U_t is the total sum of discounted utility of the young during period t. During period $t+1$, one agent expect to have $1+n$ children. Since the parents are selfish in that they care about themselves more than their children, the benefits from their children are discounted by $\frac{1}{1+\psi}$. Hence their utility from their children is $\frac{1+n}{1+\psi}U_{t+1}$.

Assume that

$$
0 = \lim_{s\to\infty} \beta^s U_{t+s}.
$$

where $\beta = \frac{(1+n)}{(1+\rho)(1+\psi)}$. Then utility of generation t is

$$
U_t = \sum_{s=0}^{\infty} \beta^s \left[\frac{C_{yt+s}^{1-\theta}-1}{1-\theta} + \frac{1}{1+\rho}\frac{C_{ot+1+s}^{1-\theta}-1}{1-\theta}\right],
$$

Homework: Derive the utility of generation t.

Assume that parents can transfer their income to their children. Then the budget constraint of each generation must change:

$$C_{yt} + S_t = w_t + m_t \tag{65}$$
$$C_{ot+1} + (1+n)\, m_{t+1} = (1+i_{t+1})\, S_t \tag{66}$$

where m_t is the transfer from the parents to their children. Hence,

$$m_{t+1} = \frac{1+i_{t+1}}{1+n}\left[w_t + m_t - C_{yt}\right] - \frac{C_{ot+1}}{1+n}$$

We can define the Bellman equation of this problem as

$$\begin{aligned} V(m_t, w_t, i_{t+1}) &= \max_{C_{yt}, C_{ot}} \left\{ \left[\frac{C_{yt}^{1-\theta} - 1}{1-\theta} + \frac{1}{1+\rho}\frac{C_{ot+1}^{1-\theta} - 1}{1-\theta} \right] + \beta V(m_{t+1}, w_{t+1}, i_{t+2}) \right\} \\ s.t.\ m_{t+1} &= \frac{1+i_{t+1}}{1+n}\left[w_t + m_t - C_{yt}\right] - \frac{C_{ot+1}}{1+n} \end{aligned}$$

The first order conditions are

$$C_{yt}^{-\theta} = \beta V_1(m_t, w_t, i_{t+1}) \frac{1+i_{t+1}}{1+n} = \frac{1+i_{t+1}}{(1+\rho)(1+\psi)} V_1(m_t, w_t, i_{t+1}) \tag{67}$$
$$C_{ot+1}^{-\theta} = \beta V_1(m_t, w_t, i_{t+1}) \frac{1+\rho}{1+n} = \frac{1}{1+\psi} V_1(m_t, w_t, i_{t+1}) \tag{68}$$

Then we can derive

$$\frac{C_{ot+1}}{C_{yt}} = \left(\frac{1+i_{t+1}}{1+\rho}\right)^{\frac{1}{\theta}} \tag{69}$$

This condition is equivalent to the first order condition of the standard OGM, [equation (55)]. Hence,

$$C_{ot+1} = \left(\frac{1+i_{t+1}}{1+\rho}\right)^{\frac{1}{\theta}} C_{yt}$$

Substituting this equation into the original problem,

$$\begin{aligned} V(m_t, w_t, i_{t+1}) &= \max_{C_{yt}} \left\{ U(C_{yt}, i_{t+1}) + \beta V(m_{t+1}, w_{t+1}, i_{t+2}) \right\} \\ s.t.\ m_{t+1} &= \frac{1+i_{t+1}}{1+n}(w_t + m_t) - p(i_{t+1})\, C_{yt} \end{aligned}$$

where

$$\begin{aligned} U(C_{yt}, i_{t+1}) &= \frac{1}{1-\theta}\left[\left[1 + \frac{(1+i_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}} \right] C_{yt}^{1-\theta} - \frac{2+\rho}{1+\rho} \right] \\ p(i_{t+1}) &= \frac{1}{1+n}\left[(1+i_{t+1}) + \left(\frac{1+i_{t+1}}{1+\rho}\right)^{\frac{1}{\theta}} \right] \end{aligned}$$

This problem is equivalent to

$$\max_{\{C_{yt+s}\}} \sum_{s=0}^{\infty} \beta^s U\left(C_{yt+s}, i_{t+s+1}\right)$$

$$s.t\ m_{t+1} = \frac{1+i_{t+1}}{1+n}\left(w_t + m_t\right) - p\left(i_{t+1}\right) C_{yt}$$

This is essentially the problem of representative agent. In fact, the budget constraint can be rewritten as

$$\sum_{s=0}^{\infty}\prod_{\tau=0}^{s+1}\left(\frac{1+n}{1+i_{t+\tau}}\right) p\left(i_{t+s+1}\right) C_{yt+s} = \sum_{s=0}^{\infty}\prod_{\tau=0}^{s}\left(\frac{1+n}{1+i_{t+\tau}}\right) w_{t+s} + m_t$$

Proof.

$$\begin{aligned}\frac{1+n}{1+i_{t+1}} m_{t+1} &= \left(w_t + m_t\right) - \frac{1+n}{1+i_{t+1}} p\left(i_{t+1}\right) C_{yt} \\ m_t &= \frac{1+n}{1+i_{t+1}} m_{t+1} + \frac{1+n}{1+i_{t+1}} p\left(i_{t+1}\right) C_{yt} - w_t\end{aligned}$$

$$\begin{aligned}m_t &= \frac{1+n}{1+i_{t+1}}\left[\frac{1+n}{1+i_{t+2}} m_{t+2} + \frac{1+n}{1+i_{t+2}} p\left(i_{t+2}\right) C_{yt+1} - w_{t+1}\right] + \frac{1+n}{1+i_{t+1}} p\left(i_{t+1}\right) C_{yt} - w_t \\ &= \frac{1+n}{1+i_{t+1}}\frac{1+n}{1+i_{t+2}} m_{t+2} + \frac{1+n}{1+i_{t+1}}\left[\frac{1+n}{1+i_{t+2}} p\left(i_{t+2}\right) C_{yt+1} - w_{t+1}\right] + C_{yt} + \frac{C_{ot+1}}{1+i_{t+1}} - w_t \\ &= \sum_{s=0}^{\infty}\prod_{\tau=0}^{s}\left(\frac{1+n}{1+i_{t+\tau}}\right)\left[\frac{1+n}{1+i_{t+s+1}} p\left(i_{t+s+1}\right) C_{yt+s} - w_{t+s}\right]\end{aligned}$$

Therefore,

$$\sum_{s=0}^{\infty}\prod_{\tau=0}^{s+1}\left(\frac{1+n}{1+i_{t+\tau}}\right) p\left(i_{t+s+1}\right) C_{yt+s} = \sum_{s=0}^{\infty}\prod_{\tau=0}^{s}\left(\frac{1+n}{1+i_{t+\tau}}\right) w_{t+s} + m_t$$

■

The left hand side is the discounted sum of the stream of consumption expenditure and the right hand side is the permanent income of generation t. Hence, the permanent income hypothesis holds. In the representative agent model, the market economy is Pareto optimal. The overlapping generation model with altruism can also attains Pareto optimal. Because parents care about children's utility and budget constraint is connected by bequests, any transfer from young to old cannot improve old's utility.

Note that this analysis implicitly assume that income transfer always occurs. When the parents are selfish enough, they do not want to leave any bequests. It happens when the first order conditions (67) or (68) are not satisfied by equality, and solution is boundary. Hence, the parents' wish to consume every their income. In this case, the intergenerational linkages are broken and the transfer from the next generation improve the utility of the current generation. Hence, the equivalence of the representative agent model and the OGM occurs if the parents' altruism are strong enough.

5 The Long Run Growth Rate:

Let me move on to more recent issues. In Modern Economics 1, I said that the long run growth rate is entirely determined by the movement of technology, T_t. This central result does not change after considering the detail of a consumer behavior. Hence, I start to turn my attention to the micro foundation of productivity growth in this section. Since the late 80's, many macroeconomists have started to endogenize productivity growth, which is called endogenous growth model. Many model has been presented. I review a few famous models among them.

Endogenous growth models are intellectually exciting. The traditional neoclassical growth model assumes that the growth rate of productivity is exogenous. Therefore, there is no policy to change the long run growth rate. Since many development economists sought to find a policy to enhance the growth rate of developing countries, this theoretical result is not attractive and there was no meaningful communication between growth theorists and development economists during 60's. Endogenous growth model made a bridge between growth theorists and development economists.

However, endogenous growth model has not been supported by evidence. I have already discussed Jones's critique in Modern Macroeconomics 1. I provides another evidence which contradicts the essence of the endogenous growth theory - convergence discussion. The traditional neoclassical growth model predicts that the growth rate must converges to the exogenous long run level of growth rate, while endogenous growth model predicts that different countries must demonstrate different growth rate depending on their policy. I discuss the empirical discussion of convergence.

First, I explain the two typical endogenous growth model. One is human capital model; the other is R&D model.

5.1 Human Capital Accumulation and the Long Run Growth

Uzawa formulates human capital growth model. Using the Uzawa's human capital growth model, Lucas (1988) analyzes development issues. The model is formulated as follows.

$$\max_{(c_t,u_t)} \int_0^\infty e^{-\rho t} \frac{(c_t T_t)^{1-\theta}}{1-\theta} dt$$

$$\begin{aligned}
s.t.\ \dot{K}_t &= K_t^\alpha \left[u_t T_t N_t\right]^{(1-\alpha)} - \delta K_t - c_t T_t N_t,\ K_0 \ is\ given \\
\dot{T}_t &= B T_t (1-u_t),\ T_0\ is\ given. \\
\dot{N}_t &= n N_t,\ N_0\ is\ given.
\end{aligned}$$

The main difference from the previous neoclassical growth model is that the growth rate of productivity, $\frac{\dot{T}_t}{T_t}$ is not exogenous. The the variable, u_t, is the amount of time to spend at work. This model assumes that the agent accumulates human capital when he does not work. Since in this model, the growth rate of technology depends on the amount of time to accumulate human capital, any policy which can influence u_t changes the long run growth rate.

As usual, we can rewrite the problem using variables with the unit of effective labor:

$$\max_{(c_t,u_t)} \int_0^\infty e^{-\rho t} \frac{(c_t T_t)^{1-\theta}}{1-\theta} dt$$

$$\begin{aligned}
s.t.\ \dot{k}_t &= \left[k_t^\alpha (u_t)^{(1-\alpha)} - c_t\right] - (\delta + B(1-u_t) + n) k_t,\ k_0\ is\ given \\
\dot{T}_t &= B T_t (1-u_t),\ T_0\ is\ given
\end{aligned}$$

Homework: Check this problem is the same as before.

Define the Hamiltonian of the above problem:

$$H = \frac{(c_t T_t)^{1-\theta}}{1-\theta} + \lambda_t \left\{ \left[k_t^\alpha (u_t)^{(1-\alpha)} - c_t\right] - (\delta + B(1-u_t) + n) k_t \right\} + \mu_t B T_t [1-u_t]$$

Hence, the first order conditions are

$$\lambda_t = (c_t T_t)^{-\theta} T_t \tag{70}$$

$$\mu_t B T_t = \lambda_t \left[(1-\alpha) k_t^\alpha (u_t)^{-\alpha} + B k_t\right] \tag{71}$$

$$\dot{\lambda}_t = \rho\lambda_t - \left[\alpha k_t^{\alpha-1} (u_t)^{(1-\alpha)} - (\delta + B(1-u_t) + n)\right] \lambda_t,\ \lim_{t\to\infty} e^{-\rho t} \lambda_t k_t = 0 \tag{72}$$

$$\dot{\mu}_t = \rho\mu_t - (c_t T_t)^{-\theta} c_t - B(1-u_t)\mu_t,\ \lim_{t\to\infty} \mu_t T_t = 0 \tag{73}$$

$$\dot{k}_t = \left[k_t^\alpha (u_t)^{(1-\alpha)} - c_t\right] - (\delta + B(1-u_t) + n) k_t,\ k_0\ is\ given \tag{74}$$

$$\dot{T}_t = B T_t (1-u_t),\ T_0\ is\ given \tag{75}$$

Assume that economy is on the steady state, where steady state is defined by $\dot{k}_t = \dot{c}_t = \dot{u}_t = 0$. Then it is shown from equation (75),

$$\frac{\dot{T}_t}{T_t} = B(1-u) = g$$

Hence, on the steady state the growth rate of productivity depends on u.

Homework: Solve the steady state value of u. Hint: From (70) and (72), derive

$$(1-\theta) B(1-u) = \rho - \left[\alpha k^{\alpha-1} (u)^{(1-\alpha)} - (\delta + B(1-u) + n)\right]$$

From equation (74), derive

$$c = k^{\alpha} (u)^{(1-\alpha)} - (\delta + B(1-u) + n) k$$

From equations (70), (71), (72), (73) and (75), derive

$$\frac{cB}{(1-\alpha) k^{\alpha} (u)^{-\alpha} + Bk} = \alpha k^{\alpha-1} (u)^{(1-\alpha)} - (\delta + B(1-u) + n)$$

These are three equations and three unknown, k, c, u. Hence you can solve it.

5.2 R&D Model

Romer (1990) develops the model of innovation. Since knowledge is nonrival, it takes long time to produce new knowledge. However, once new idea is produced, we can easily imitate. It means that innovation needs a huge fixed cost, but the marginal cost is small. Hence, it is likely that the average cost is declining and the production function demonstrates increasing return to scale. Hence, if the market is competitive, a firm can not make any profits, and nobody makes any effort on innovation. To give a firm an incentive to innovate, the firm's idea must be protected by law so that the firm can enjoy the monopoly rent from the new idea.

Consumer: A representative consumer solves the same problem as before:

$$\begin{aligned} & \int_0^{\infty} e^{-\rho t} \frac{\left(\frac{C_t}{N_t}\right)^{1-\theta} - 1}{1-\theta} dt \\ \dot{A}_t &= i_t A_t + w_t N_t - C_t, \ A_0 \ is \ given \end{aligned}$$

Hence, the following Euler equation describes the growth rate of consumption: (Check it by yourself.)

$$\begin{aligned} \frac{\dot{C}_t}{C_t} &= \frac{1}{\theta} \left[i_t - \rho - (1-\theta) n\right] \\ 0 &= \lim_{t \to \infty} e^{-[\rho + (1-\theta) n] t} (C_t)^{-\theta} A_t \end{aligned}$$

Final Goods Sector and Intermediate Goods Sector: Romer model is a three sector model: the final goods sector, the intermediate goods sector and research sector. Let me first describe the final goods sector and the intermediate goods sector. As the neoclassical growth model, the final goods sector consists of the perfectly competitive firms, which combine labor and capital, and produce a homogeneous output good, Y_t. The main difference from the traditional aggregate production function is that they employ more than one capital good, which is called intermediate goods:

$$Y_t = \left(s_t^f N_t\right)^{1-\alpha} \int_0^{T_t} x_t(j)^\alpha \, dj \tag{76}$$

where $x_t(j)$ are the jth intermediate good, s_t^f is the share of workers who work at the final goods sector. In this model, the variable, T_t, measures the variety of the intermediate goods, which are available to the production of the final goods at date t. Hence, the final goods sector solves

$$\begin{aligned} \pi_t^f &= \max_{s_t^f, x_t(i)} \left\{ Y_t - w_t s_t^f N_t - \int_0^{T_t} p_t(j)\, x_t(j)\, dj \right\}, \\ s.t.\ Y_t &= \left(s_t^f N_t\right)^{1-\alpha} \int_0^{T_t} x_t(j)^\alpha \, dj. \end{aligned}$$

The first order conditions are

$$w_t = (1-\alpha)\left(s_t^f N_t\right)^{-\alpha} \int_0^{T_t} x_t(j)^\alpha \, dj, \tag{77}$$

$$p_t(j) = \alpha \left(s_t^f N_t\right)^{(1-\alpha)} x_t(j)^{\alpha-1}, \ for\ \forall j. \tag{78}$$

The intermediate goods sector consists of monopolists who produce the capital goods that are sold to the final goods sector. It rents capital goods, $\kappa_t(j)$, from the market and produces the intermediate goods, $x_t(j)$. For the simple explanation, I assume that the production function is $\kappa_t(j) = x_t(j)$. Hence, the intermediate firms maximizes its profits given the demand curve for the goods and the rental price of the capital stock:

$$\pi_t(j) = \max_{x_t(j)} \left\{ p_t(j)\, x_t(j) - r_t \kappa_t(j) \right\}, \tag{79}$$

$$\begin{aligned} s.t.\ p_t(j) &= \alpha \left(s_t^f N_t\right)^{(1-\alpha)} x_t(j)^{\alpha-1}. \\ x_t(j) &= \kappa_t(j) \end{aligned}$$

where r_t is the rental price of capital. The first order condition is

$$\begin{aligned} r_t &= \alpha^2 \left(s_t^f N_t\right)^{(1-\alpha)} x_t(j)^{\alpha-1} \\ &= \alpha p_t(j) \end{aligned}$$

Hence $x_t(j) = x_t$ and $p_t(j) = p_t$. Therefore, it is derived from equations (76), (77), (78) and (79) that

$$p_t = \frac{r_t}{\alpha} \tag{80}$$

$$\pi_t(j) = \pi_t = (1-\alpha)\frac{r_t}{\alpha}x_t \tag{81}$$

$$Y_t = \left(s_t^f N_t\right)^{1-\alpha} x_t^\alpha T_t \tag{82}$$

$$w_t = (1-\alpha)\left(s_t^f N_t\right)^{-\alpha} x_t^\alpha T_t \tag{83}$$

Equation (80) implies that the price of the intermediate good is greater than the marginal cost of production (= the rental price) since $\alpha < 1$. Since the intermediate goods sector has the monopoly power, it can sell its products higher than its marginal cost. Therefore, as equation (81) shows, the intermediate goods sector yields profits.

Define the aggregate capital stock, K_t, is the sum of the capital stock demanded by the intermediate goods sector: :

$$K_t = \int_0^{T_t} \kappa_t(j)\, dj = \int_0^{T_t} x_t(j)\, dj = x_t T_t. \tag{84}$$

Then substituting this definition into equations (78), (80), (82) and (83), I can derive

$$Y_t = (K_t)^\alpha \left(T_t s_t^f N_t\right)^{1-\alpha}, \tag{85}$$

$$w_t = (1-\alpha)(K_t)^\alpha \left(s_t^f N_t T_t\right)^{-\alpha} T_t = (1-\alpha)\frac{Y_t}{s_t^f N_t}. \tag{86}$$

$$\frac{r_t}{\alpha} = \alpha\left(s_t^f N_t T_t\right)^{(1-\alpha)} (K_t)^{\alpha-1} = \alpha\frac{Y_t}{K_t} \tag{87}$$

Note that the production function at the final goods sector [= equation (85)] is consistent with the aggregate production function in the neoclassical growth model. In this model, when the variety of the intermediate goods increases, the productivity of the aggregate production function increases. Equations (86) and (87) correspond to the first order conditions of the firm's profit maximization problems under the neoclassical growth model. The main difference from the neoclassical growth model is that the marginal product of capital stock is not equal to the rental price, r_t, but $\frac{r_t}{\alpha}$ [See equation (87)]. Because the intermediate goods sector has the monopoly power, it sets a monopoly price for intermediate goods. Therefore, the marginal cost of employing capital is higher than the rental price.

Since the marginal cost of employing capital is larger than the rental price, intermediate goods sector earns monopoly rent. To see this, substituting equation (84), (85) and (87) into equation (81), I can derive profits of the intermediate goods sector:

$$\pi_t = (1-\alpha)\,\alpha \frac{Y_t}{T_t}, \tag{88}$$

Using equations (85), (86) (87) and (88),

$$Y_t = w_t s_t^f N_t + r_t K_t + \pi_t T_t$$

This equation shows that the value of production is distributed into labor expense, the rental cost of capital and the monopoly profits of the intermediate sector. Because the intermediate sector has the monopoly power, the marginal product of capital stock is shared by the final goods sector and the intermediate goods sector.

This monopoly profits gives the research sector an incentive to invent new idea and to establish new firms. The new idea invented by research sector increases the variety of the intermediate goods and the productivity in this model.

Research Sectors: Assume that a research sector employs researchers and invent new idea for intermediate goods:

$$\dot{T}_t = D_t s_t^T N_t,$$

where s_t^T is the share of workers which is allocated to the research sector. This production function implies that the more researchers, the more likely to find new idea. Since idea is nonrival, there is externality from old idea to create new idea. I assume that

$$D_t = BT_t^{\beta},\ \beta \leq 1.$$

Hence, the knowledge accumulation equation is

$$\dot{T}_t = BT_t^{\beta} s_t^T N_t$$

Once the research sector invent new idea, it can sell this idea and earn the market value of the new idea, P_t^T. Hence, the research sector solves the following problem.

$$\begin{aligned} \pi_t^R &= \max_{L_t} \left\{ P_t^T \dot{T}_t - w_t s_t^T N_t \right\}, \\ s.t.\ \dot{T}_t &= BT_t^{\beta} s_t^T N_t, \end{aligned}$$

where P_t^T is the market value of new idea. Hence the first order condition of the research sector is

$$\beta P_t^T BT_t^{\beta} = w_t.$$

Arbitrage Conditions: Once somebody buy new idea, he can establish a new intermediate firm. If he does so, he expects to receive profits, π_t, every period, and can obtain capital gain, $\dot{P}_t^T$. Hence, the return from owing the firm is $\frac{\pi_t + \dot{P}_t^T}{P_t^T}$. On the other hand, he can sell his idea to somebody else, put money into bank and earn the interest rate; $i_t P_t^I$. Hence the return from investing riskless asset i_t. If the financial market is competitive, two returns must be equivalent:

$$\frac{\pi_t + \dot{P}_t^T}{P_t^T} = i_t$$

Similarly, if consumers invest in physical capital, the return is $r_t - \delta$. It has to be equal to the return from investing riskless asset, i_t:

$$r_t - \delta = i_t$$

Labor Market Clearing Condition: the labor market clearing condition implies that the demand for labor is equal to the supply of labor. Hence, total population must be allocated into either the final goods sector or the research sector:

$$s_t^f + s_t^T = 1$$

Capital Market Clearing Condition: the consumers can invest in either physical capital or stocks of the intermediate goods sector. Hence supply of assets, A_t, is equal to the sum of physical capital and the market value of the intermediate goods sector:

$$A_t = K_t + P_t^T T_t$$

Equilibrium: Let me define the market equilibrium.

Definition 5 *The market economy consists of 10 variables* $\left\{C_t, A_t, Y_t, K_t, w_t, r_t, \pi_t, i_t, P_t^T, s_t^f, s_t^T, T_t\right\}$ *which satisfy the following conditions:*

1. *Consumers' maximization problem determines* C_t *and* A_t*:*

$$\begin{aligned} \frac{\dot{C}_t}{C_t} &= \frac{1}{\theta}\left[i_t - \rho - (1-\theta)\, n\right],\ 0 = \lim_{t\to\infty} e^{-[\rho+(1-\theta)n]t} \left(C_t\right)^{-\theta} A_t \\ \dot{A}_t &= i_t A_t + w_t N_t - C_t,\ A_0 \text{ is given} \end{aligned}$$

2. *The final sectors' and intermediate sectors' maximization problems determine* Y_t*,* s_t^f, K_t *and* π_t*:*

$$\begin{aligned} Y_t &= \left(K_t\right)^{\alpha} \left(s_t^f N_t T_t\right)^{1-\alpha} \\ w_t &= (1-\alpha)\frac{Y_t}{s_t^f N_t} \\ r_t &= \alpha^2 \frac{Y_t}{K_t} \\ \pi_t &= (1-\alpha)\,\alpha \frac{Y_t}{T_t} \end{aligned}$$

3. The research sector determines s_t^T, T_t:

$$\dot{T}_t = BT_t^{\beta} s_t^T N_t$$

$$BP_t^T T_t^{\beta} = w_t$$

4. Two arbitrage conditions determine P_t^T and r_t:

$$\frac{\pi_t + \dot{P}_t^T}{P_t^T} = i_t$$

$$r_t - \delta = i_t$$

5. Labor market clearing condition determines w_t:

$$s_t^f + s_t^T = 1$$

6. Capital market clearing condition determines i_t:

$$A_t = K_t + P_t^T T_t$$

Homework: Derive the following three equations which summarize the behavior of our economy given s_t^T.

$$\begin{aligned} \dot{K}_t &= (K_t)^{\alpha} \left[\left(1 - s_t^T\right) N_t T_t\right]^{1-\alpha} - \delta K_t - C_t \\ \frac{\dot{C}_t}{C_t} &= \frac{1}{\theta} \left[\alpha^2 \left[\left(1 - s_t^T\right) N_t T_t\right]^{(1-\alpha)} (K_t)^{\alpha-1} - \rho - \delta - (1-\theta) n\right] \\ \dot{T}_t &= BT_t^{\beta} s_t^T N_t \end{aligned}$$

Hint: review how do we derive the capital accumulation equation under the representative agent model. I plug several equations into the consumers' budget constraint. You can apply the same method to derive capital accumulation equation.

To solve this problem

$$\begin{aligned}
\dot{A}_t + C_t &= i_t A_t + w_t N_t \\
\dot{A}_t + C_t &= i_t \left(K_t + P_t^T T_t\right) + w_t s_t^f N_t + w_t s_t^T N_t \\
&= (r_t - \delta) K_t + i_t P_t^T T_t + (1-\alpha)\frac{Y_t}{s_t^f N_t} s_t^f N_t + P_t^T B T_t^\beta s_t^T N_t \\
&= \alpha^2 \frac{Y_t}{K_t} K_t - \delta K_t + \left(\pi_t + \dot{P}_t^T\right) T_t + (1-\alpha) Y_t + P_t^T \dot{T}_t \\
&= \alpha^2 Y_t - \delta K_t + \left((1-\alpha)\alpha \frac{Y_t}{T_t} + \dot{P}_t^T\right) T_t + (1-\alpha) Y_t + P_t^T \dot{T}_t \\
&= Y_t - \delta K_t + \frac{dP_t^T T_t}{dt} \\
\dot{K}_t + \frac{dP_t^T T_t}{dt} + C_t &= Y_t - \delta K_t + \frac{dP_t^T T_t}{dt} \\
\dot{K}_t &= Y_t - C_t - \delta K_t
\end{aligned}$$

Homework: Show that the above three equation can be written as the unit of effective labor as follows:

$$\dot{k}_t = (k_t)^\alpha \left(1 - s_t^T\right)^{1-\alpha} - \left(\delta + n + g_t^T\right) k_t - c_t \quad (89)$$

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}\left[\alpha^2 \left(1 - s_t^T\right)^{(1-\alpha)} (k_t)^{\alpha-1} - \rho - \delta - n - g_t^T \theta\right] \quad (90)$$

$$g_t^T = B T_t^{\beta-1} s_t^T N_t \quad (91)$$

where $k_t = \frac{K_t}{T_t N_t}$, $c_t = \frac{C_t}{T_t N_t}$ and $n = \frac{\dot{N}_t}{N_t}$.

Note that if s_t^T and g_t^T is constant, equations (89) and (90) are the similar to two fundamental equations of the neoclassical growth model. The main difference is that the first term of the Euler equation in the representative agent model is $\alpha\left(1 - s_t^T\right)^{(1-\alpha)} (k_t)^{\alpha-1}$; in this model $\alpha^2\left(1 - s_t^T\right)^{(1-\alpha)} (k_t)^{\alpha-1}$. Since the intermediate goods sector has the monopoly power, Hence, we know that the long rung growth rate only depends on g_t^T, which is determined by equation (91). Note also that three equations are similar to Uzawa-Lucas model, when $\beta = 1$ and N_t is constant. In fact, in the Romer's original model, $\beta = 1$ and N is constant:

$$g_t^T = B s_t^T N$$

In this case, when the government increases the number of researchers, the government can raise the long run growth rate.

As I discussed in Modern Macroeconomics 1, Jones (1995) criticizes this assumption: although we observe an increase in the number of researchers in the US, the growth rate is fairly constant. Following Jones suggestion, I assume $\beta < 1$.

Homework: Review the essence of Jones's critique.

Let me analyze three equations at the steady state. On the steady state, $\dot{c}_t = 0$, $\dot{k}_t = 0$, $g_t^T = g$, $s_t^T = s^T$ and $\dot{P}_t = 0$.

$$\begin{aligned} c &= (k)^\alpha \left(1 - s^T\right)^{1-\alpha} - (\delta + n + g) k \\ \alpha^2 \left(1 - s^T\right)^{(1-\alpha)} (k)^{\alpha - 1} &= \rho + \delta + n + g\theta \\ g &= \frac{n}{1-\beta} \end{aligned}$$

Hence, the long run growth rate is $\frac{n}{1-\beta}$ and as long as we assume population growth rate is constant, government can not change the long run growth rate. In this model, the long run growth rate is endogenous in that it is explicitly modeled. However, it is exogenous in that government can not manipulate the long rung growth rate.

In order to find the steady state level of output, I need to find the steady state level of s^T. I can show that

$$s^T = \frac{n\alpha}{(1-\beta)\rho + n(1+\theta+\alpha-\beta)}.$$

Proof.

$$\begin{aligned} \frac{(1-\alpha) y^* T_t}{(1-s^T)} &= BT_t^\beta \frac{\pi_t}{i_t} \\ &= BT_t^\beta \frac{(1-\alpha)\alpha \frac{Y_t}{T_t}}{\alpha^2 (1-s^T)^{(1-\alpha)} (k^*)^{\alpha-1} - \delta} \\ &= \frac{BT_t^\beta N_t (1-\alpha)\alpha y^*}{\rho + n + g\theta} \\ &= \frac{gT_t (1-\alpha)\alpha y^*}{s^T (\rho + n + g\theta)} \end{aligned}$$

Hence

$$\frac{1}{(1-s^T)} = \frac{g\alpha}{s^T(\rho+n+g\theta)}$$

Hence

$$\begin{aligned} s^T &= \frac{g\alpha}{\rho + n + g(\theta+\alpha)} \\ &= \frac{n\alpha}{(1-\beta)\rho + n(1+\theta+\alpha-\beta)} \end{aligned}$$

■

The market economy in the Romer's model is not Pareto optimum. There are two distortions in the model. Firstly, the intermediate sector has the monopoly power. It raises the marginal cost of capital stock and discourages employing new capital. Secondly, there is positive externality from old invention to new one. Researchers are not compensated for their contribution toward improving the productivity of future research. Therefore, there is little research than social optimum.

Homework: Solve the following social planner's problem and show that the market economy is not social optimum.

$$\int_0^{\infty} e^{-\rho t} \frac{\left(\frac{C_t}{N_t}\right)^{1-\theta} - 1}{1-\theta} dt$$

$$\begin{aligned}
\dot{K}_t &= (K_t)^{\alpha} \left[\left(1 - s_t^T\right) N_t T_t\right]^{1-\alpha} - \delta K_t - C_t \\
\dot{T}_t &= B T_t^{\beta} s_t^T N_t
\end{aligned}$$

5.3 A simple Model of the Diffusion

Although R&D model is interesting, innovation may not be the issue for the most of the country. The better technology is already given for developing countries. In fact, many empirical research suggests that R&D has little impact on the productivity growth. Jovanovic argues that even in developed countries, innovation is not the main source of growth, but the adoption of new technology is important.

Following Jones (2002), I propose a simple model of diffusion in this section. Social planner solves the following problem:

$$\max_{(c_t)} \int_0^{\infty} e^{-\rho t} \frac{(c_t T_t)^{1-\theta} - 1}{1-\theta} dt$$

$$\begin{aligned}
s.t.\ \dot{k}_t &= k_t^{\alpha} - c_t - \left(\delta + \frac{\dot{T}_t}{T_t} + n\right) k_t,\ k_0 \text{ is given} \\
\dot{T}_t &= B \left(T_t^F\right)^{\beta} T_t^{(1-\beta)},\ T_0 \text{ is given} \\
\dot{T}_t^F &= g T_t^F,\ T_0^F \text{ is given}
\end{aligned}$$

where T_t^F is the frontier technology, which grows by the constant rate, g. In this model, it is assumed that the larger the distance between the productivity of the frontier technology and of developing countries is, the higher the growth rate of technology adoption.

$$\frac{\dot{T}_t}{T_t} = B \gamma_t^{\beta}.$$

where $\gamma_t = \frac{T_t^F}{T_t}$. That is, the relative position to the frontier technology, $\frac{1}{\gamma_t}$ is low, the growth rate of productivity is high. This relationship corresponds to the argument of the benefit of the relative backwardness. Developing countries can attain high growth exploiting the benefit of relative backwardness. Economic historians, also argue that developing countries need social capability to exploit the benefit. The productivity parameter B can be interpreted as the measure of social capability in this model.

Homework: Show that given T_t^F the economy of the developing countries can be summarized by the following two equations:

$$\begin{aligned} \frac{\dot{c}_t}{c_t} &= \frac{1}{\theta}\left[\alpha k_t^{\alpha-1} - \left(\rho + \delta + n + \theta B \gamma_t^{\beta}\right)\right] \\ \dot{k}_t &= k_t^{\alpha} - c_t - \left(\delta + B\gamma_t^{\beta} + n\right) k_t \end{aligned}$$

Consider the steady state, that is, c_t, k_t and γ_t is constant. Then

$$\begin{aligned} \alpha \left(k^*\right)^{\alpha-1} &= \rho + \delta + n + \theta B \left(\gamma^*\right)^{\beta} \\ c^* &= \left(k^*\right)^{\alpha} - \left(\delta + B\left(\gamma^*\right)^{\beta} + n\right) k^* \end{aligned}$$

On the steady state, $\gamma_t = \frac{T_t^F}{T_t}$ is constant. Hence,

$$\frac{\dot{T}_t}{T_t} = \frac{\dot{T}_t^F}{T_t^F} = g.$$

That is, the long run growth rate of the developing countries in this model is the same as the long run growth rate of the technology frontier. This prediction is consistent with some of the development facts, which was discussed in Modern Macroeconomics 1: Income distributions across countries shifts up, and the relative income differences do not show any convergence. However, it contradicts one of the growth facts and one of the development facts: the growth rates across countries differ, and there is economic miracle and economic disaster.

How can we reconcile the prediction of this model and these evidence? One possibility is that most of countries are still on the transition process. Several factors can force economy to temporarily deviate from the steady state. For example, when government lowers capital income tax rate, it encourages saving and, therefore, investment. It temporarily raises the growth rate of the economy. A temporal deviation from the long run trend may be able to explain different growth rate.

Let me derive the steady state output. Using the usual technique, it is easy to show that

$$\frac{Y_t}{N_t} = \left[\frac{\alpha}{\rho + \delta + n + \theta g}\right]^{\frac{\alpha}{1-\alpha}} \frac{T_t^F}{\gamma^*}$$

Homework: Derive the above equation: GDP per capita.

Note that

$$g = \frac{\dot{T}_t}{T_t} = B\left(\gamma^*\right)^{\beta}$$

Hence

$$\gamma^* = \left[\frac{g}{B}\right]^{\frac{1}{\beta}}$$

Hence, I can derive the following GDP per capita on the steady state.

$$\frac{Y_t}{N_t} = \left[\frac{\alpha}{\rho+\delta+n+\theta g}\right]^{\frac{\alpha}{1-\alpha}}\left(\frac{B}{g}\right)^{\frac{1}{\beta}} T_0^F e^{gt} \tag{92}$$

This is the similar to the one by the representative agent model. The main difference is the productivity parameter B can affect the level of GDP per capita.

There are three factors affecting B: general skill, technology specific skill and the barrier to technology adoption. Nelson and Phelps (1966) argue that education helps adopting new technology. Welch (1970) provides evidence that education enhances the ability to adopt new technology. Shultz (1976) call this ability entrepreneurial ability. Hence, entrepreneurial ability in a society is large, it is easier to adopt new technology.

Parente (1994) emphasizes the importance of technology specific skill for the adoption of new technology. When a firm adopt new technology, if there is technology specific skill, they must destroy skills for old technology. Hence, the opportunity cost of adopting new technology is high.

Another issue related to a technology specific skill is the dynamic efficiency vs. the static efficiency. Krugman (198?) argues that it explains the story of Holland. Holland had the comparative advantage to the agriculture. Hence, many people in Holland engage into an agricultural sector, and they could not accumulate skill at the manufacturing sector. Hence, Holland is behind with industrialization. Young (198?) argues that the free trade may take away the opportunity to learn technology specific skill through learning by doing, since the developing country can import such products. These arguments support the protection of an infant industry: if an industry is infant and government protect the industry, they get the opportunity to learn a skill specific to the industry.

The last argument is that there might have the barrier to adopt new technology. Parente and Prescott (1994) argues that there is the barrier to adopt new technology and this barrier can explain a huge income differences across countries. In England, skilled labor has opposed to the adoption of new economy. Other developing countries corruption can bring such a barrier. These barriers can lower the parameter B.

5.4 Convergence

I have sketched a micro foundation of an exogenous growth model in that government can not change the long run growth rate. It provides a theoretical justification for an assumption, the long run growth rate of different country can be the same, g. In this view, the difference in growth rates across countries can be seen as the transition process to the long run growth rate. Hence, this model predicts convergence toward the steady state. On the other hand, different prediction was made by endogenous growth models, which endogenous means that a policy change can affect the long run growth rate. Since a change in policy has an impact on the long run growth rate, as far as different countries adopt different policies, they can not converge to the same growth rate. This has brought a big discussion among economists. I review these discussions.

Consider the original Solow model with the constant saving rate:

$$\dot{k}_t = s\left(k_t\right)^{\alpha} - \left(g + n + \delta\right)k_t$$

Note that the constant saving rate is justified on the steady state of the representative agent model. We have already know that the following equation must be satisfied on the steady state:

$$s\left(k^*\right)^{\alpha} = \left(g + n + \delta\right)k^*$$

I assume that most of countries locate near the steady state, and derive the growth rate by a log linear approximation around the steady state.

Homework: Review Taylor approximation and show that a log linear approximation of a function $f\left(k\right)$ around k^* has a following form:

$$f\left(k\right) = f\left(k^*\right) + f'\left(k^*\right)k^*\left(\log k - \log k^*\right)$$

Absolute β convergence: Using the neoclassical growth model, Barro and Sala-i-Martin (1992) proposed the concept of β convergence. Their definition of β convergence is as follows: If a poor economy tends to grow faster than a rich one, β convergence occurs. I first derive empirically testable equation which identifies β

convergence.

$$\begin{aligned}
\frac{d\log y_t}{dt} &= \frac{\dot{y}_t}{y_t} \\
&= \frac{\alpha k_t^{\alpha-1}\dot{k}_t}{(k_t)^{\alpha}} \\
&= \alpha\frac{\dot{k}_t}{k_t} \\
&= \alpha\left[s\left(k_t\right)^{\alpha-1}-(g+n+\delta)\right] \\
&= \alpha\left[s\left(\exp\left(\log k_t\right)\right)^{\alpha-1}-(g+n+\delta)\right] \\
&\cong \alpha s\left(\alpha-1\right)\left(k^*\right)^{a-2}k^*\left(\log k_t-\log k^*\right) \\
&= -\left(1-\alpha\right)s\left(k^*\right)^{\alpha-1}\left(\log y_t-\log y^*\right) \\
&= -\beta\left(\log y_t-\log y^*\right) \\
\beta &= \left(1-\alpha\right)\left(g+n+\delta\right)
\end{aligned}$$

since $s\left(k_t^*\right)^{\alpha}=\left(g+n+\delta\right)k_t^*$. Since $\frac{d\log y^*}{dt}=0$, I can modify the above equation as

$$\begin{aligned}
\frac{d\left(\log y_t-\log y^*\right)}{dt} &= -\beta\left(\log y_t-\log y^*\right) \\
\log y_t-\log y^* &= \left(\log y_0-\log y^*\right)e^{-\beta t} \\
\log y_t-\log y_0 &= \left(1-e^{-\beta t}\right)\left(\log y^*-\log y_0\right)
\end{aligned}$$

Barro and Sala-i-Martin (1992) assumed that every region in the united state converges to the same steady state. That is, every region has the same y^*. Then

$$\begin{aligned}
&\log\left(\frac{Y_t}{L_t}\right)-\log\left(\frac{Y_0}{L_0}\right) \\
=\ & \log T_t-\log T_0+\left(1-e^{-\beta t}\right)\left(\log y^*-\log\frac{Y_0}{L_0}+\log T_0\right) \\
\cong\ & gt+\left(1-e^{-\beta t}\right)\log y^*T_0-\left(1-e^{-\beta t}\right)\log\frac{Y_0}{L_0}+\varepsilon
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{\log\left(\frac{Y_t}{L_t}\right)-\log\left(\frac{Y_0}{L_0}\right)}{t} \\
&= a-\frac{\left(1-e^{-\beta t}\right)}{t}\log\frac{Y_0}{L_0}+\varepsilon \\
whegre\ a &= g+\frac{\left(1-e^{-\beta t}\right)}{t}\log y^*T_0
\end{aligned}$$

If β convergence occurs, β should be positive. In this case, the growth rate is a negative function of the initial income per capita. Using regional data in the United State, Barro and Sala-i-Martin (1992) estimated $\beta = 0.02$. Hence, they find the evidence of β convergence. Since $\beta = (1-\alpha)(g+n+\delta)$, I can recover α from this β. An implied α is equal to 0.8. As I discussed in Modern Macroeconomics 1, reasonable number is around 0.3. Too large. Remember that if α is much larger than 0.3, the neoclassical growth model might be able to explain a large income differences across country using the neoclassical growth model.

Homework: Review discussions about income differences in Modern Macroeconomics 1.

They also conduct the same regression using cross country data. They got negative sign. It means that there is no evidence of β convergence.

Conditional β convergence: The main prediction of the neoclassical growth model is that every country should converges to its own steady state, but not to the same steady state. If the steady state differs, the absolute convergence is not predicted by the model.

$$
\begin{aligned}
\log y_t - \log y_0 &= \left(1-e^{-\beta t}\right)\left(\log y^* - \log y_0\right) \\
&= \left(1-e^{-\beta t}\right)\left(\log (k^*)^\alpha - \log y_0\right) \\
&= \left(1-e^{-\beta t}\right)\left(\log \left(\frac{s}{g+n+\delta}\right)^{\frac{\alpha}{1-\alpha}} - \log y_0\right)
\end{aligned}
$$

$$
\begin{aligned}
& \log\left(\frac{Y_t}{L_t}\right) - \log\left(\frac{Y_0}{L_0}\right) \\
= & \frac{\alpha\left(1-e^{-\beta t}\right)}{1-\alpha}\log s - \frac{\alpha\left(1-e^{-\beta t}\right)}{1-\alpha}\log(g+n+\delta) \\
& -\left(1-e^{-\beta t}\right)\log\left(\frac{Y_0}{L_0}\right) + \left(1-e^{-\beta t}\right)T(0) + \left(\log T_t - \log T_0\right) \\
\cong & \frac{\alpha\left(1-e^{-\beta t}\right)}{1-\alpha}\log s - \frac{\alpha\left(1-e^{-\beta t}\right)}{1-\alpha}\log(g+n+\delta) \\
& -\left(1-e^{-\beta t}\right)\log\left(\frac{Y_0}{L_0}\right) + \left(1-e^{-\beta t}\right)T(0) + gt + \varepsilon
\end{aligned}
$$

When Mankiw, Romer and Weil (1992) did cross-section regression by assuming $T(0)$ is the same across countries, they found the evidence of conditional convergence. But it has too small β and, therefore, too large α.

Modified neoclassical growth model and conditional β convergence: Note that the neoclassical growth model predicts the correct direction, the saving rate has a positive impact on GDP per capita, and population growth has a negative impact on the GDP per capita. However, the estimated α of the production function, k_t^α is too large. Since α is capital share, it has to be around 1/3. One of possibility is that there is unmeasured capital and the real capital share is larger than 0.3. In order to investigate this possibility, Mankiw, Romer and Weil (1992) add one more capital in their model: human capital:

$$\begin{aligned} \dot{K}_t &= s_k Y_t - \delta K_t \\ \dot{H}_t &= s_h Y_t - \delta H_t \\ Y_t &= K_t^\alpha H_t^\psi (T_t N_t)^{(1-\alpha-\psi)} \end{aligned}$$

where, s_k is the share of income investing physical capital, s_h is the share of income investing human capital, $\frac{\dot{T}_t}{T_t} = g$, and $\frac{\dot{N}_t}{N_t} = n$. This notion of human capital differs from Uzawa-Lucas Model. Human capital in Mankiw, Romer and Weil (1992) is formulated as additional capital stock, and not the component of T_t. Note that they assume that physical capital stock and human capital has the same depreciation rate. Using the unite of effective labor, I can express above equation as

$$\begin{aligned} \dot{k}_t &= s_k k_t^\alpha h_t^\psi - (n+g+\delta) k_t, \\ \dot{h}_t &= s_h k_t^\alpha h_t^\psi - (n+g+\delta) h_t, \end{aligned}$$

where $k_t = \frac{K_t}{T_t N_t}$, $y_t = \frac{Y_t}{T_t N_t}$ and $h_t = \frac{H_t}{T_t N_t}$. Hence, on the steady state,

$$\begin{aligned} s_k (k^*)^\alpha (h^*)^\psi &= (n+g+\delta) k^*, \\ s_h (k^*)^\alpha (h^*)^\psi &= (n+g+\delta) h^*. \end{aligned}$$

Hence on the steady state, k^* and h^* has a proportional relationship.

$$k^* = \frac{s_k}{s_h} h^*$$

Using this relationship, I can derive the steady state value of k^* and h^*:

$$\begin{aligned} s_h \left(\frac{s_k}{s_h} h^* \right)^\alpha (h^*)^\psi &= (n+g+\delta) h^* \\ (h^*)^{1-\alpha-\psi} &= \frac{(s_k)^\alpha (s_h)^{1-\alpha}}{n+g+\delta} \\ h^* &= \left[\cdot \frac{(s_k)^\alpha (s_h)^{1-\alpha}}{n+g+\delta} \right]^{\frac{1}{1-\alpha-\psi}} \end{aligned}$$

$$
\begin{aligned}
k^* &= \frac{s_k}{s_h}\left[.\frac{(s_k)^{\alpha}(s_h)^{1-\alpha}}{n+g+\delta}\right]^{\frac{1}{1-\alpha-\psi}} \\
&= \left[.\frac{(s_k)^{1-\psi}(s_h)^{\psi}}{n+g+\delta}\right]^{\frac{1}{1-\alpha-\psi}}
\end{aligned}
$$

Hence on the steady state, GDP per capita is

$$
\begin{aligned}
\frac{Y}{L} &= \left[.\frac{(s_k)^{1-\psi}(s_h)^{\psi}}{n+g+\delta}\right]^{\frac{\alpha}{1-\alpha-\psi}}\left[.\frac{(s_k)^{\alpha}(s_h)^{1-\alpha}}{n+g+\delta}\right]^{\frac{\psi}{1-\alpha-\psi}} T_t \\
&= .\frac{(s_k)^{\frac{\alpha}{1-\alpha-\psi}}(s_h)^{\frac{\psi}{1-\alpha-\psi}}}{(n+g+\delta)^{\frac{\alpha+\psi}{1-\alpha-\psi}}} T_t
\end{aligned}
$$

Derivation of Convergence Equation: Let me applying the same excises to MRW model

$$
\begin{aligned}
\frac{d\log y_t}{dt} &= \frac{\dot{y}_t}{y_t} \\
&= \frac{\alpha k_t^{\alpha-1}h_t^{\psi}\dot{k}_t}{(k_t)^{\alpha}h_t^{\psi}} + \frac{\beta k_t^{\alpha}h_t^{\psi-1}\dot{h}_t}{(k_t)^{\alpha}h_t^{\psi}} \\
&= \frac{\alpha\dot{k}_t}{k_t} + \frac{\beta\dot{h}_t}{h_t} \\
&= \alpha s_k (k_t)^{\alpha-1} h_t^{\psi} + \beta s_h (k_t)^{\alpha} h_t^{\psi-1} - (\alpha+\beta)(g+n+\delta) \\
&= \left[\alpha(\alpha-1) s_k (k^*)^{\alpha-1}(h^*)^{\psi} + \alpha\psi s_h (k^*)^{\alpha}(h^*)^{\psi-1}\right][\log k_t - \log k^*] \\
&\quad + \left[\alpha\psi s_k (k^*)^{\alpha-1}(h^*)^{\psi} + \psi(\psi-1) s_h (k^*)^{\alpha}(h^*)^{\psi-1}\right][\log h_t - \log h^*] \\
&= [\alpha(\alpha-1)(g+n+\delta) + \alpha\psi(g+n+\delta)][\log k_t - \log k^*] \\
&\quad + [\alpha\psi(g+n+\delta) + \psi(\psi-1)(g+n+\delta)][\log h_t - \log h^*] \\
&= -(1-\alpha-\psi)(g+n+\delta)[\log(k_t)^{\alpha} - \log(k^*)^{\alpha}] \\
&\quad -(1-\alpha-\psi)(g+n+\delta)\left[\log(h_t)^{\psi} - \log(h^*)^{\psi}\right] \\
&= -(1-\alpha-\psi)(g+n+\delta)(\log y_t - \log y^*) \\
\textit{where } \beta &= (1-\alpha-\psi)(g+n+\delta)
\end{aligned}
$$

I used the steady state conditions: $s_k (k^*)^{\alpha}(h^*)^{\psi} = (g+n+\delta)k$ and $s_h (k^*)^{\alpha}(h^*)^{\psi} = (g+n+\delta)h$ in the above derivation. Since this is the same equation as before, I can derive $\log y_t - \log y_0 = \left(1-e^{-\beta t}\right)(\log y^* - \log y_0)$ as before. Hence, using the steady state condition, I can derive equation for conditional convergence.

$$\begin{aligned}
\log y_t - \log y_0 &= \left(1-e^{-\beta t}\right)\left(\log y^* - \log y_0\right) \\
&= \left(1-e^{-\beta t}\right)\left(\log\left[\frac{(s_k)^{\frac{\alpha}{1-\alpha-\psi}}(s_h)^{\frac{\psi}{1-\alpha-\psi}}}{(n+g+\delta)^{\frac{\alpha+\psi}{1-\alpha-\psi}}}\right] - \log y_0\right) \\
&= \frac{\alpha\left(1-e^{-\beta t}\right)}{1-\alpha-\psi}\log(s_k) + \frac{\psi\left(1-e^{-\beta t}\right)}{1-\alpha-\psi}\log(s_h) \\
&\quad -\frac{(\alpha+\psi)\left(1-e^{-\beta t}\right)}{1-\alpha-\psi}\log(n+g+\delta) - \left(1-e^{-\beta t}\right)\log y_0
\end{aligned}$$

Hence

$$\begin{aligned}
\log\left(\frac{Y_t}{L_t}\right) - \log\left(\frac{Y_0}{L_0}\right) &\cong a + \frac{\alpha\left(1-e^{-\beta t}\right)}{1-\alpha-\psi}\log(s_k) + \frac{\psi\left(1-e^{-\beta t}\right)}{1-\alpha-\psi}\log(s_h) \\
&\quad -\frac{(\alpha+\psi)\left(1-e^{-\beta t}\right)}{1-\alpha-\psi}\log(n+g+\delta) \\
&\quad -\left(1-e^{-\beta t}\right)\log\left(\frac{Y_0}{L_0}\right) + \varepsilon
\end{aligned}$$

If this model is correct, there are four reasons the original Solow model produces large α:

1. If s_h and s_k are positively correlated, then the coefficient of s_k in the original regression would be upper biased.

2. If n and s_h are negatively correlated, then the coefficient of n in the original regression would be lower biased.

3. Even though s_h and s_k are not correlated, an implied α would be smaller in the modified model since $\frac{\alpha}{1-\alpha-\psi} > \frac{\alpha}{1-\alpha}$ and ψ would be positive.

4. Even though s_h and n are not correlated, an implied α would be smaller in the modified model since $\frac{\alpha+\psi}{1-\alpha-\psi} > \frac{\alpha}{1-\alpha}$ and ψ would be positive.

Data: Summers and Heston dataset. Non-Oil countries, Non-countries except for grade D countries and small population countries, OECD countries.

1. Y/L...real GDP in 1985 divided by the working age population in that period.

2. n... the average growth of working age population over 1960-1985, where working age is defined as 15 to 64.

3. s_k...the average share of real investment in real GDP over 1960-1985.

4. s_h...[The fraction of the eligible population (aged 12 to 17) enrolled in the secondary school]x[the fraction of working age population that is of school age (aged 15 to 19)].

5. $g + \delta$...They assume $g + \delta = 0.05$.

They show that s_h and s_k is positively correlated and s_h and s_n is negatively correlated: $\rho_{s_h s_k} = 0.59$. $\rho_{s_h n} = -0.38$. This evidence support their above conjecture.

Results: Mankiw, Romer and Weil (1992) find the following empirical results.

1. Evidence of conditional convergence

2. The restriction, the sum of coefficients of $\log s_h$, $\log s_k$ and $\log(n + g + \delta)$, is equal to 0, can not be rejected.

3. The slow convergence $\beta = 0.02$.

4. An implied α is about 0.4, and an implied ψ is about 0.23.

As you can see, the implied α is close to 1/3. This result can be considered as a big success of the neoclassical growth model by Mankiw, Romer and Weil.

Omitted Variable Bias and Panel Approach: T_0 is most likely correlated with y_0, s_k, s_h and n. Since ε in Mankiw, Romer and Weil (1992) model include the variation of T_0, the results will be biased. When Islam (1995) did the same regression with panel data set, he allowed different $T(0)$ by using country dummy variable. He found about β is about 0.04 and implied α is about 0.5. Moreover, he found that human capital measure is not significant and sometimes shows wrong sign. It indicates a omitted variable bias.

Islam's result and Mankiw, Romer and Weil's result provides different economic interpretations of evidence. In Mankiw, Romer and Weil's model, once we control human capital, convergence occurs. That is, the main difference of income across countries come from human capital. However, Islam's result indicate country dummy is important. It indicates that productivity difference across countries are the main factor to explain income difference.

Criticism against convergence method:
Galton's Fallacy: Quah (1993) criticized convergence regressions by showing the following example. Assume that cross section distribution is the stationary for many

independent and identical distribution of country income $\log\left(\frac{Y_t}{L_t}\right)$. For β convergence regression means

$$\begin{aligned}
\log\left(\frac{Y_t}{L_t}\right) &= E\left(\log\left(\frac{Y_t}{L_t}\right)\right)+\lambda\left(\log\left(\frac{Y_0}{L_0}\right)-E\left(\log\left(\frac{Y_0}{L_0}\right)\right)\right)+\varepsilon \\
where\ \lambda &= \frac{Cov\left(\log\left(\frac{Y_t}{L_t}\right),\log\left(\frac{Y_0}{L_0}\right)\right)}{Var\left(\log\left(\frac{Y_0}{L_0}\right)\right)}
\end{aligned}$$

But Cauchy Schwarz inequality implies

$$\begin{aligned}
\left|Cov\left(\log\left(\frac{Y_t}{L_t}\right),\log\left(\frac{Y_0}{L_0}\right)\right)\right| &\leq \left\{Var\left(\log\left(\frac{Y_0}{L_0}\right)\right)\right\}^{\frac{1}{2}}\left\{Var\left(\log\left(\frac{Y_1}{L_1}\right)\right)\right\}^{\frac{1}{2}} \\
&= Var\left(\log\left(\frac{Y_0}{L_0}\right)\right)
\end{aligned}$$

Hence λ is always less than 1. Hence

$$\log\left(\frac{Y_t}{L_t}\right)-\log\left(\frac{Y_0}{L_0}\right)=c-(1-\lambda)\log\left(\frac{Y_0}{L_0}\right)+\varepsilon$$

This regression shows $(1-\lambda)\geq 0$, no matter what. Since we can see β convergence. It simply implies the following intuition. Suppose that you observe two realizations sequentially from i.i.d. distribution, and suppose that the first observation is quite small. Then probability which the second observation is larger than the first observation is high. Hence, even though distribution is i.i.d., we will observe β convergence.

σ convergence and β convergence. By following Quah's criticism, Barro and Sala-i-Martin (1995) clarified the difference between β convergence and σ convergence as follows. Barro and Sala-i-Martin (1995) defined

- If a poor economy tends to grow faster than a rich one, β convergence occurs.
- If the standard deviation of the logarithm of per capita income declines over time, σ convergence occurs.

This is different. Suppose that following is true.

$$\log\left(\frac{Y_t}{L_t}\right)=c-\lambda\log\left(\frac{Y_{t-1}}{L_{t-1}}\right)+\varepsilon$$

where $0 < \lambda < 1$ and $\varepsilon \tilde{} N(0, \sigma)$. Then

$$\begin{aligned} \sigma^2_{\log yt} &= \lambda^2 \sigma^2_{\log yt-1} + \sigma^2, \\ &= \lambda^2 \left(\sigma^2_{\log yt-1} - \frac{\sigma^2}{1-\lambda^2} \right) + \frac{\sigma^2}{1-\lambda^2} \\ &= .\lambda^{2t} \left(\sigma^2_{\log y0} - \frac{\sigma^2}{1-\lambda^2} \right) + \frac{\sigma^2}{1-\lambda^2} \end{aligned}$$

Hence an initial $\sigma^2_{\log y0}$ is greater than $\frac{\sigma^2}{1-\lambda^2}$, then $\sigma^2_{\log yt}$ shrinks. If an initial $\sigma^2_{\log y0}$ is less than $\frac{\sigma^2}{1-\lambda^2}$, then $\sigma^2_{\log yt}$ increases. β convergence cause tendency to reduce $\sigma_{\log yt}$, but since we can always observe a random shock which is not related to convergence, this shock may increase $\sigma_{\log yt}$.

Distribution Dynamics: Durlauf and Quah (1998) still argued that σ convergence can not capture mobility in the distribution. And they propose an alternative method. Separate countries into several groups by income level, say 2 groups: rich and poor (relative to US). Take two years, say 1960 and 1985. Compute probabilities that rich in 1960 stays rich in 1985, that rich becomes poor, that poor becomes rich, that poor stays poor. This provides a transition dynamics of distributions. They assume that this dynamics is stable, and analyze the behavior of distributions.

This approach has an advantage of directly looking at the movement of the whole distribution. In fact, Quah (1993) finds that relative income distribution across countries shows two peaks, which I discussed in modern macroeconomics 1. He said that his evidence supports convergence in club. However, there is a disadvantage: Since there is no structural model behind this approach, we can apply Lucas's critique: new policy changes estimates of distribution dynamics. Hence, it may not be robust.

5.5 My current View

The above empirical discussion suggests that we need to carefully define the word, convergence. Note, however, that convergence in club is still consistent with conditional convergence. Hence, the best available evidence supports the prediction of the exogenous growth model; countries converge to their own steady state. Together with Jones's critique, there is little evidence which supports an endogenous growth model. (The endogenous growth model means the model under which a policy change can influence the long run growth rate.)

Of course, endogenous growth models provide many insights into economic growth. Although data shows that the long run growth rate is fairly constant despite the fact that there are many policy changes, practically speaking, it may not be productive to separate the long run growth rate from the short run growth rate. Since it is easier to analyze economy on the steady state than that on the transition path, if it takes more 30 years to reach the steady state, it might not be bad idea to analyze the impact of a policy change on the growth rate by using the endogenous growth model. In sum, I view that the current development of endogenous growth models help illustrating qualitative insights, but these models can not be a foundation of quantitative investigation.

If we wish to construct an endogenous growth model as a foundation of quantitative investigation, I think that we should answer why industrialization occurs. We know that the long run growth rate of the US economy in these 200 years is fairly constant. However, if the US economy experienced the constant growth rate since the beginning of human history, per capita consumption must be less than subsistence level at some points. The best available evidence suggests that we started the long run growth some day around industrialization. The main question should be why and how the long run growth started. It would answer the source of long run growth rate. That is why many macroeconomists try to construct a model of industrialization.

Finally, note that β convergence does not tell anything about the mechanism of convergence. The neoclassical growth theory predicts the convergence to the steady state, because the marginal productivity of capital declines. On the other hand, the model of diffusion which I has discussed before shows a different channel: the convergence through the diffusion of knowledge. Originally economic historian supported this channel, though growth theorists do not pay enough attention to diffusion. Notion of β convergence can not distinguish two diffusion processes.

6 Productivity Slowdown and Vintage Capital Model

The story of technology progress is invention and subsequent implementation of improved methods of production. However, the exogenous growth model assumes that the productivity rains down as manna from heaven. Productivity Growth measured by Solow residual $R(t)$, where

$$R(t) = g_{\frac{Y}{N}} - \left(1 - \frac{wL}{Y}\right) g_{\frac{K}{N}},$$

became small after 1970s. This evidence is odd because we observe new technology after 1970s. It gives a question what is productivity and what Solow residual captures. That is, Solow (1957) model cannot explain why productivity slow downs after 1970s. There is another facts that the Solow (1957) model cannot explain: the falling price of capital goods relative to consumption goods. The relative price of equipment falls by about 4 % in the U.S..

In this section, we develop vintage capital model and ask how vintage capital model can explain above evidence. Solow (1957) is that it treats all vintages of capital as alike. In reality, advances in technology tend to be embodied in the latest vintages of capital. This means that new capital is better than old capital. It also means that there can be no technological progress without investment. We start with Solow (1960) model as the baseline vintage model.

Assume $T_t = 1$ and redefine that $y_t = \frac{Y_t}{N_t}$, $k_t = \frac{K_t}{N_t}$, $i_t = \frac{I_t}{N_t}$ and $s_t = \frac{S_t}{N_t}$. Let us consider the following dynamics

$$\begin{aligned} i_t &= s_t = sf(k_t) \\ \dot{k}_t &= q_t i_t - (n+\delta) k_t \\ \dot{q}_t &= g q_t \end{aligned}$$

where k_0 and q_0 are given. Different from the standard model, this model assumes that one unit of output is transformed to produce $q_t i_t$ unit of capital. It implicitly means that the investment firm solves the following profit maximization problem.

$$\begin{aligned} &\max \{p_t k_t^n - i_t\} \\ s.t.\ k_t^n &= q_t i_t \end{aligned}$$

where p_t is the price of capital goods relative to consumption goods at date t and k_t^n is newly product capital at date t. The solution to this problem is

$$p_t q_t = 1$$

Hence, the productivity growth embodied in capital goods can be estimated by

$$q_t = \frac{1}{p_t}.$$

It means that the declining price of equipment indicates an improvement in the productivity of equipment.

$$g_q = -g_p$$

Because the price drops by 4%, the estimated improvement in the productivity of equipment is 4%. Hence, there is no slowdown in improvement in q. Combining two equations, we can derive the following two dynamics.

$$\begin{aligned} \dot{k}_t &= s q_t f(k_t) - (n+\delta) k_t \\ \dot{q}_t &= g q_t \end{aligned}$$

where k_0 and q_0 are given. The main difference from the neoclassical growth model is the improvement in q_t.

This model is not perfect because it means the further reduction of the measured Solow residual. Note that

$$R(t) = g_y - \left(1 - \frac{wN}{Y}\right) g_k$$

Hence, the growth rate of per capita capital is large, the measured Solow residual is low. Current model means that

$$k_t = \int_0^\infty q_{t-a} i_{t-a} e^{-(n+\delta)a} da.$$

Standard model assumes $q_t = 1$, but q_t grows by 4%. Hence, g_k is larger in the current model. Therefore, the slowdown of the TFP growth is larger than the estimate before.

In order to explain the productivity slowdown puzzle, Jovanovic and Greenwood (2001) argues that the vintage capital model with learning and diffusion lags can explain productivity slowdown puzzle.

- **Learning Effect:** They argue that productivity can temporarily fall upon a switch to a new technology because a new technology may be operated inefficiently. They show several evidence on learning effects.

    1. David (1975) undertook a case study on Lawrence no.2 cotton mill. He documented that no new equipment was added between 1836 and 1856, but output per hour grew at 2.3 percent per year over the period.
    2. Bahk and Gort (1993) investigate 2000 firms from forty one industries between 1973 and 86. They find that a plant's productivity increases by 15 percent over the first fourteen years of its life due to learning effect.

- **Diffusion Lags:** They also note that the diffusion of innovation is slow. Gort and Klepper (1982) examined that forty-six product innovations, beginning with phonograph records in 1887 and ending with lasers in 1960s. Using the same data set, Jovanovic and Lach (1997) shows that it took approximately fifteen years from the output of a new product to rise from the 10 percent to the 90 percent diffusion level. There are several theories for the diffusion lags.

    1. *Vintage Specific Physical Capital:* New equipment is costly, while a firm has already bought old ones. Hence it is optimal to wait before replacing old equipment.
    2. *Vintage Specific Human Capital:* Because a firm with old technology has employed old skilled workers, new young workers can get benefits from old skilled workers. Hence, young workers may hesitate to adopt new technology and new business.
    3. *Second-Mover Advantages:* The experience of early adopters is of help to those that adopt later. Hence, firms have an incentive to delay.
    4. *Lack of Awareness:* A firm may not be aware of any or all of the following: that (a) a new technology exists, (b) that it is suitable, or where to acquire all the complementary goods.

6.1 Incomplete Contract, Specific Investment, Diffusion Lags and Unemployment

There is another important reason that the diffusion of new technology is slow: hold up problem. Hold up problem arises when investment is relation-specific and the cost of investment cannot be written in contract. When investment is technology specific, the adoption of technology may delay, but it is an optimal behavior. When investment is relation-specific, it is possible that the investment is lower than optimal ones. When an agent makes investment used only with other parties and the cost of investment cannot be written in a contract, other parties can always walk away without any cost. Hence, other parties can threaten the investor and investor cannot be able to receive all the return on investment.

Long run relationship is often associated with either switching costs or specific investment. When there is switching costs or specific investment, staying together can yield a surplus relative to trading with other parties. A crucial aspect of specific investment is that even though the supplier and the buyer may select each other ex ante in a pool of competitive suppliers and buyers, they end up forming an ex post bilateral monopoly in that they have incentive to trade between them rather than with outside parties.

Consider the following two period model: t=1 or 2. A supplier and a buyer try to trade a good which is suitable to a specific demand of a buyer. You can interpret a supplier as a worker, and a buyer as a firm. Both parties are risk neutral. At the first period, a worker invests to improve human capital, h and and yields output. Then a firm will pay wage, w. The firm can sell the output at a price 1. Since human capital is useful only for this firm, if they fire the worker, worker do not get anything.

Human capital investment incurs a cost $C(h)$. Assume that $C'(h) > 0$ and $C''(h) > 0$. The two parties realize the level of investment and the level of cost at the beginning of the second period. The key assumption is that two parties can not make a contract at the first period. That is because

1. both parties may not be able to foresee contingent future, or
2. even though they can foresee, they may not be able to describe the contingent future, or
3. even though they can describe, writing every contingency is quite costly.

Both party can observe the level of investment at the first period, but it is not verifiable. Investment depends not only on money to spend but also a supplier's effort. Then it is difficult to verify the effort in front of court. So they can not make a contract on it. Even though they can not contract on the level of investment, if they can describe cost precisely at the first period, they may be able to make a contingent contract based on the level of cost. But it is often difficult to describe the possible cost precisely. Cost may include the depreciation of a machine, but it is difficult to estimate the level of depreciation without doing anything.

The first best: Before analyzing this problem, let me describe what is the first best investment. The first best investment maximizes expected net benefit

$$\max_{I} \{h - C(h)\},$$

For a simple analysis, let me assume that an agent does not discount at all. Hence the first order condition is

$$1 = C'\left(h^{best}\right).$$

The second period problem: At the second period, everything becomes clear. A firm and a worker may be able to negotiate the wage w. Assume that if workers quite a job worker can get W and if a firm fires the worker, a firm can get P. We assume that wage can be determined to share the surplus: $h - W - P$. Assume that the bargaining power of workers is assumed to be $\beta \in (0,1)$. Then the wage bargaining determine

$$w = W + \beta(h - W - P)$$

The first period: A worker knows that the price will be determined as above. Given this knowledge he decides his investment decision at period 1. His problem is

$$\begin{aligned} W &= \max_{h} [w - C(h)], \\ s.t.\ w &= W + \beta(h - W - P) \end{aligned}$$

The first order condition is

$$\beta = C'(h^*)$$

Because $\beta < 1$, and $C'' > 0$

$$h^* < h^{best}.$$

That is, a worker does underinvestment. The reason is that since investment is specific to the firm, after making investment, the investment is sunk. If the firm fire the worker, investment is useless. Bargaining over wage reduces the marginal benefit from the investment. Since he knows it will happen, it discourage his investment. He will optimally reduce his investment. This is called "Hold-up problem." The main problem here is that the party investing does not capture all the cost savings generated by his investment. The other party can use the thread of not trading to appropriate some of these savings.

The above analysis suggest that an increase in β increases human capital accumulation. If $\beta = 1$, $h^* = h^{best}$. Because a worker is a person to make investment decision, a larger bargaining power of worker encourages human capital accumulation. Because h^* is lower than social optimal value, larger human capital is welfare improving. So it is good to provide more power on workers.

Diffusion Lags and Unemployment: I would like to apply the above argument into the problem of diffusion. For simple explanation, I approximate the story by a static model. I show that it naturally causes diffusion lags. Moreover, it may also cause involuntary unemployment. For this purpose, I take completely an opposite position. A firm pays a training cost and worker does not do anything for human capital investment. This is the simpler version of Caballero (2007). A firm buy new equipment by P and train a worker by T and produces A output. Training cost is sunk But after the training, the worker can walk away. Then the firm keeps equipment and must find a new worker. The worker knows this consequence and threaten the firm. If the worker walks away, the worker can find the similar job with probability $\frac{E}{L}$ where E is the number of employed workers and L is the number of labor force. With probability $\left(1-\frac{E}{L}\right)$, the worker cannot find a similar job and receives unemployment benefits or wage from the secondary market. This reservation value is denoted by z. Then the surplus from this much is

$$S = A - \left[\frac{E}{L}w + \left(1-\frac{E}{L}\right)z\right] - P \tag{93}$$

where w is the wage paid by the similar job. Assume that the bargaining power of workers, the wage is determined by

$$w = \beta S + \frac{E}{L}w + \left(1-\frac{E}{L}\right)z \tag{94}$$

It shows that if the worker decides to leave, the worker can expected to get $\frac{E}{L}w + \left(1-\frac{E}{L}\right)z$. This is the threat point of workers. Addition to these values, the worker can receive β part of surplus. On the other hand, the firm expected to receive

$$J = (1-\beta)S + P \tag{95}$$

If the worker leaves, the firm keeps the market value of equipment P. Hence, the firm's outside option is P. Addition to this value, the firm can receive $(1-\beta)$ portion of surplus from this match.

Before the production, the firms buy new equipment and find and train workers. The training cost is denoted by T. The free entry condition implies that

$$T + P = J \tag{96}$$

This completes equilibrium. Equations (93), (94), (95) and (96) determines $\left(S, w, J, \frac{E}{L}\right)$. Let me solve the model. The equation (95) is substituted into the free entry condition

(96).

$$
\begin{aligned}
T+P &= (1-\beta)\,S+P \\
T &= (1-\beta)\,S \\
S &= \frac{T}{1-\beta}
\end{aligned}
\tag{97}
$$

Hence, the surplus is determined by training cost and retirement allowance.

Now, substituting equation (94) and (97) into the definition of surplus (93),

$$
\begin{aligned}
S &= A-\left[\frac{E}{L}w+\left(1-\frac{E}{L}\right)z\right]-P \\
S &= A-[w-\beta S]-P \\
(1-\beta)\,S &= A-w-P \\
w &= A-(1-\beta)\,S-P \\
&= A-T-P
\end{aligned}
\tag{98}
$$

The equation implies that the wage is independent of the bargaining power. When the bargaining power is large, because the worker can receive the large share of surplus, the wage is large. On the other hand, if the bargaining power is small, many firms are reluctant to enter the market because they cannot receive much surplus. It reduces the number of jobs to be offered and the number of employed workers. Hence, the workers find difficulty find goods job and lowers worker's threat point. In this model, two opposite effects are always cancelled out and w is independent of β.

If the bargaining power does not change wages, what is the economic consequence of bargaining power. Substituting equations (97) and (98) into equation (94), we can derive that

$$
\begin{aligned}
w &= \beta S+\frac{E}{L}w+\left(1-\frac{E}{L}\right)z \\
\frac{E}{L}\,(w-z) &= w-z-\beta S \\
\frac{E}{L} &= 1-\frac{\beta S}{w-z} \\
&= 1-\frac{\beta\frac{T}{1-\beta}}{A-T-P-z} \\
&= 1-\frac{\beta T}{(1-\beta)\,(A-T-P-z)}
\end{aligned}
$$

It shows that large β, T and z lowers $\frac{E}{L}$. That is, all parameters that helps workers lower employment probability. Because of the hold up problem, the offered job is the smaller than optimal. When $\beta=0$ or $T=0$, the firm can receive all the benefits from training, $\frac{E}{L} = 1$. That is, if there is no opportunistic behavior or no specific investment, the firms employ all workers who want to work in this economy. Because the number of jobs is fewer than labor force in the equilibrium, this indicates that investment in new technology is not immediate. It shows the existence of diffusion lags.

Alternatively, $\frac{E}{L} < 1$ also indicates the existence of unemployed workers.

$$\frac{U}{L} = \frac{\beta T}{(1-\beta)(A-T-P-z)}$$

This unemployment is involuntary because they want to work if they can. Finally per capita GDP is also the function of $\frac{E}{L}$.

$$\frac{Y}{L} = \frac{AE}{L} = A\left[1 - \frac{\beta T}{(1-\beta)(A-T-P-z)}\right]$$

Because we have unused resources, involuntary unemployed workers, clearly this economy is inefficient.

Remarks: Clearly this is an extreme view. As I have shown before, once we introduce the worker's effort to accumulate human capital, an opposite story can be possible. If so, real question would be which agents are more contributing to invest in a firm specific human capital.

Recent cross-country evidence shows that job creation rates are not significantly different across countries having different labor market policies where job creation rate is defined as follows.

$$\begin{aligned} GJCR_t &= \frac{\sum_{i\in I^+} |E_{it} - E_{it-1}|}{N_t}, I^+ = \{i | E_{it} \geq E_{it-1}\} \\ N_t &= \sum_{i\in I} \frac{E_{it} + E_{it-1}}{2} \end{aligned}$$

where E_{it} is the number of workers employed in ith plant at year t. This is inconsistent with the provided theory. It indicates that the reality is more complex. Instead, literature finds that the worker flows are between 1.5 and 2.5 times larger in the United States than in Europe, where worker flows are defined as

$$GWFR_t = \frac{\sum_{i\in I}(H_{it} + S_{it})}{N_t}$$

where H_{it} and S_{it} are the number of workers hired by and separated from the ith plant in the year t, respectively. Pries and Rogerson (2005) provides an alternative model. They argue that the quality of a worker-firm match is both an inspection good and an experience good. At the time of meeting, both parties have limited information about the match's quality, which is completely revealed only by engaging in production. The worker and firm forms a match when the expected value of match is high and are separated when realized match is low. Labor regulation can influences these decisions. Because labor regulation in the Europe is much restrictive than that in the U.S., the firms in the Europe selectively employ workers and maintain the relationship. On the other hand, firms in the U.S. less selectively employ workers and fire them if bad match is realized. The exercises in Pries and Rogerson (2005) suggest that an increase in the minimum wage and high dismissal costs have significant impact on worker flows.

7 Income Differences

If countries converge to their own steady state, then the next natural question is why different countries have different steady state. The neoclassical growth model provides us only partial answers to this question: the different saving rate and the growth rate of population. The modified neoclassical growth model adds human capital investment to the candidates.

Mankiw, Romer and Weil (1992) claims that if we add human capital to the neoclassical growth model, we can explain the major parts of income differences across countries. If they are right, the next research agenda is to investigate the source of the different saving rate, population growth and human capital.

However, as I said in the previous chapter, Islam (1995) find that once he controls country dummy in his regression, human capital measure is not significant and sometimes shows wrong sign. It indicates human capital is not the main cause of income differences, and there are other factors affecting productivity. Islam (1995) is not only one. As I discussed in Modern Macroeconomics 1, recently many researchers report evidence questioning the importance of human capital.

Homework: Review evidence questioning the importance of human capital, which was discussed in Modern Macroeconomics 1.

If human and physical capital is not the main source of income differences, we need to search different sources. Since the neoclassical growth model does not tell anything about alternative sources, Prescott (1999) says that the neoclassical growth model can not be the model of development. As I discuss in modern macro economics 1, many macroeconomists examines macro impacts of the institutional arrangement to enhance productive activities and prevent unproductive (=rent-seeking) activities. In particular, two effects are mainly concerned: the misallocation of talent, and the resistance to new technology.

Homework: Review discussions of the institutional arrangement, which was covered in Modern Macroeonomics 1.

Acemoglu and Ziliboth (2001) provides an alternative idea: the mismatch between technology and human capital as a source of low productivity in developing countries. Although macroeconomists speculate several possibilities, it is difficult to pin down reasonable candidates of productivity differences. Too find more reliable evidence, I review research based on microdata.

Micro Evidence on Productivity: Recently, plant level panel data is available and many researchers investigates the behaviors of plants. Bartelsman and Doms (2000) surveys evidence on productivity from longitudinal microdata. They summarizes some stylized facts:

1. The amount of productivity dispersion across plants are extremely large.
2. Productivity differences are persistent, although there is a fair amount of change in the productivity distribution.
3. A large proportion of aggregate productivity growth can be explained by resource reallocation.

To see the magnitude of the third effect, Foster, Haltiwanger and Krizan (1998) analyze the following equation. Firstly, they define the aggregate productivity growth as a weighted average of plant-level productivity growth,

$$\begin{aligned} \Delta \log T_t &= \sum s_{it} \Delta \log A_{ti} \\ \log A_{ti} &= \log Y_{ti} - \alpha_K \log K_{ti} - \alpha_L \log L_{ti} - \alpha_M \log M_{ti} \end{aligned}$$

where Y_{ti}, K_{ti}, L_{ti} and M_{ti} are output, capital input, labor input and real materials of ith plant at date t, α_K, α_L and α_M are factor shares of each input. The value s_{it} is the output share of the ith plant. Secondly, the aggregate productivity growth is decomposed into five components:

$$\begin{aligned} \Delta \log T_t = & \sum_{i \in C} s_{it-1} \Delta \log A_{ti} + \sum_{i \in C} \left(\log A_{t-1i} - \log T_{t-1}\right) \Delta s_{ti} + \sum_{i \in C} \Delta \log A_{ti} \Delta s_{ti} \\ & + \sum_{i \in N} s_{ti} \left(\log A_{ti} - \log T_{t-1}\right) - \sum_{i \in X} s_{t-1i} \left(\log A_{t-1i} - \log T_{t-1}\right) \end{aligned}$$

where C, N and X denote continuing plants, entering plants and exiting plants. The first term reflects a within-plant effect, the second term a between plant effect. The third term examines a product effect: whether activity shares shift towards plants with relatively rapid productivity growth. The last two terms capture the contribution of entering and exiting plants, respectively.

Using the Census of Manufactures in 1977 and 1987, they find that the contribution of the within plant effect, the between plant effect, the product effect and the net entry effect is 0.48, -0.08, 0.34 and 0.26. Hence, together with the product effect and net entry effect, resource reallocation explain roughly 60 % of the aggregate productivity growth.

Evidence from developing countries: The most of plant level analysis is conducted by data in the US. However, in order to answer income differences across countries, we need to seek evidence from developing countries. Tybout (2000) provides the survey of plant level analyses in developing countries.

Manufacturers in developing countries have been relatively protected. They have also been subject to heavy regulation, much of which is biased in favor of large enterprises. Hence, it is often argued that manufactures in these countries perform poorly: there are many inefficient firms, a few firms enjoy the monopoly power and many small firms are unable to grow.

However, Tybout (2000) claim that the existing empirical literature does not support that the manufacturing sector in less developed countries is inefficient. Turnover rates in plants and jobs are at least as high as those found in the OECD, and exiting plants are less productive than continuing plants. Based on his survey, Tybout (2000) speculate that the main problems in developing countries are not the monopoly power, but(1) political instability, (2) poor development of law and (3) corruption.

Resource Allocation and Learning: The emphasis on the importance of resource allocation on aggregate productivity looks contrast the views held by many Japanese labor economists: a long term relationship is necessary to accumulate a firm specific, or a relation specific skill. Koike is a leading labor economist who emphasize the importance of a long term relationship in a firm. He says that workers always faces uncertainty and unroutine works at their workplace, and that they need to accumulate an intellectual skill to deal with these unroutine works. He insists that they can accumulate such a skill only by on the job training and that it takes long time. It requires that workers do not move across firms.

I have two remarks. Firstly, the within plant effect is still large in their estimates: 0.48. The within plant effect might be explained by learning by doing effect. Secondly, the product effects do not imply that a market should be flexible. It implies that a firm which has high growth rate of productivity increases its share. Hence, two views can be consistent.

Factors affecting patterns: Bartelsman and Doms (2000) also reviews literature examining factors behind the patterns of productivity growth. They identifies four important candidates - regulation, management/ownership, technology and human capital, and international exposure, though they also mention that no candidates explain a significant proportion of the heterogeneity. Following their four candidates, I review several papers.

1. Regulation: Hoppenhayn and Rogerson investigate firm level data. They consider a tax on job destruction and find that a tax equal to 1 year's wage results in a decrease in average productivity of over 2 percent. Olley and Pakes (1996) estimate firms' exit and investment behavior in the telecommunications equipment industry and find that aggregate productivity grows faster after deregulation. They also find that most of productivity growth in the industry is due to reallocation of capital towards more productive plant.

2. International trade: many research has found that a positive relationship between exporting and productivity in a cross section data. But a problem is causality: export enhances productivity, while productive firms are likely to export. Bernard and Jensen (1999) find that relatively productive firms are likely to export, but there is little change in productivity after they start export. Doms and Jensen (1998) find that foreign owned manufacturing plants have higher total factor productivity than domestically owned plant. These evidence support the hypothesis that productive firms are likely to export. Tybout (2000) reviews plant level evidence in developing countries and find the similar evidence in developing countries. He also consider another popular theory among developing countries: protection of infant industry and concludes that evidence for fostering growth by protecting learning intensive sector is week.

3. Manager/Ownership/Organization: Lichtenberg and Siegel (1987) found (1) that a low level of TFP increases the likelihood of ownership change, and (2) that there has been improvement in the TFP of manufacturing plants after changes in corporate ownership. They interpret their results using a matching model between a manager and plant: if a match is bad, then TFP is low. This will attract ownership change. The new match will most likely be better than the previous match. On the other hand, Mcguckin and Nguyen (1995) found (1) that ownership change is generally associated with the transfer of plants with above average TFP and (2) that transferred plants experience improvements in productivity. They interpret their results using a synergy theory. If both firms have some complementary input, there is incentive to merge. Both Mcguckin and Nguyen (1995) and Lichtenberg and Siegel (1987) found a positive effect of ownership change on TFP improvement. This implies that ownership change allocates entrepreneurial talent to a more suitable position. But there is little consistency in results on the productivity of firms before ownership. Mcguckin and Nguyen (1995) argued that the different results come from the use of different datasets. Actually, the Data in Lichtenberg and Siegel (1987) covered

mainly very large firms; Mcguckin and Nguyen (1995) covered all sizes of firms. In fact, Mcguckin and Nguyen (1995) found a negative correlation between initial TFP level and the likelihood of ownership change when they restrict their dataset to only large firms. Although managers are agents who organize firms, some knowledge will be specific to the firm. Then a change in ownership can not detect the importance of this knowledge. Atkeson and Kehoe (2002) estimates organization capital - organization specific knowledge that is built up with experience. They find that roughly 4 % of output can be explained by as payments to organization capital.

4. Technology: A plenty of evidence shows the positive correlation between technology and productivity at micro level. However, causality is difficult issue. Doms, Dunne and Troske (1997) find that plants used advanced technologies in 1988 also had above average productivity in 1972. Since technology is complementary to several factors like human capital and organization, it is possible that firms with high human capital and better organization adopt advanced technology. Although there is difficulty identifying its impact, little economists doubt the importance of advanced technology on productivity. Jorgenson (199?) emphasizes the importance of quality of inputs; human capital and technology. When he carefully takes into account the quality of inputs, he finds that the TFP growth is not big component of output growth. Wolff (1996) also find that embodied technological change explains the slowdown of productivity that began in the early 1970.

7.1 My current view

Without more research, I can not identify the major factors affecting productivity. However, micro evidence gives us several hints. Combining macro speculation and micro evidence, I simply propose my tentative views. I view that the diffusion of knowledge is the main source of the growth. The diffusion process involves two procedures: the adoption of new technology and learning the best use of the technology. Productivity difference occurs due to several difficulties adopting and learning new technology. What would be the source of difficulty? I propose two possible directions of the future research.

1. *Organization Capital:* Evidence shows the main component of productivity difference across plants is unobserved and persistent. Hence, it must have an organizational specific component which affect productivity. Organization Capital - organization specific knowledge that is built up with experience is the first candidate. Since it is difficult to change organizational structure, when the adoption of new technology requires a change in organizational structure, it can cause a big barrier to adopt new one. Moreover, since this is an organization specific, it is difficult to transfer from one country to another. Evidence also

shows that entrepreneurial or managerial ability is the important component of TFP. Since entrepreneurs and managers are considered to be the persons to develop organization capital, this evidence backs up this hypothesis.

2. *Regulation and Rent-Seeking Activity:* Evidence shows that reallocation of resources is the important component of aggregate productivity. In particular, reallocation of resources toward a plant with new technology might be important. In fact, evidence shows that there is positive correlation between technology and productivity. Evidence also shows that unnecessary regulation causes entry barrier and harm productivity, and that there are more regulations in developing countries. This consideration gives us the second candidate: unnecessary regulation. Given this view, export oriented policies can be seen as the policies to bring the competition into a local market. Some regulations are necessary when there is monopoly power or externality. However, unnecessary regulations are likely to have a relation with several rent-seeking activities. The governance system in politics is an important problem.